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dirigée par Prof. Samir ADLY

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À mes parents,

à ma soeur et mes frères,

à toute ma famille.

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Le Ba Khiet

Résumé des Travaux

Ce travail concerne l'étude de la stabilité des systèmes dynamiques non réguliers. Il est basé sur l'utilisation des outils d'optimisation et d'analyse non-différentiable (non-lisse). Nous commencerons donc par donner des éléments de motivation, et ensuite, après une brève note historique sur la théorie de la stabilité, nous proposerons une introduction aux systèmes dynamiques non-réguliers, ainsi qu'un aperçu de la litérature sur le sujet concerné. Dans la dernière section, nous présenterons l'objectif et la structure de la thèse.

0.1 Motivation

Les systèmes dynamiques sont des modèles mathématiques des phénomènes évoluant dans le temps, ces phénomènes pouvant provenir de la physique, de la mécanique, de l'économie, de la biologie, de l'écologie, de la chimie, du génie électrique, etc, ... Compte tenu de leurs nombreuses applications, les systèmes dynamiques ont fait l'objet depuis le XIXe siècle d'une étude approfondie faisant intervenir différentes branches des mathématiques. La théorie est par conséquent largement étudiée et possède une position incontestablement centrale dans de nombreuses branches des sciences appliquées. Dans ce travail, nous nous intéresserons plus particulièrement à la théorie de la stabilité dont l'objectif est d'analyser le comportement d'un système dynamique sans calculer explicitement ses trajectoires solutions. L'étude de la stabilité d'un système est une exigence fondamentale pour son utilisation pratique car les systèmes dynamiques instables sont potentiellement dangereux.

Les ingénieurs et les scientifiques sont souvent confrontés à de plus en plus de modèles complexes qui ne peuvent être décrits par les outils d'analyse classique. Ceci conduit à l'apparition et au développement notable des systèmes dynamiques non-réguliers (SDNR) ainsi qu'au développement des outils d'analyse non-différentiable appropriées pour leur étude. Les SDNR existent dans toutes les branches scientifiques. Par exemple, en mécanique, les phénomènes adhérence-glissement avec frottement sec et/ou impact se formulent sous la forme de SDNR. Ils sont également présents dans les circuits électriques, les systèmes de contrôle, ainsi que dans divers modèles de l'économie, de la finance et de la biologie.... On utilise généralement des modèles réguliers pour approximer les SDNR. Néammoins ces modèles réguliers ne peuvent pas fournir des prévisions appropriées de la réalité. Par conséquent, les modèles non-réguliers doivent être mieux compris afin d'avoir une prédiction adéquate et réelle. C'est la raison pour laquelle les scientifiques du monde entier développent continûment de nouveaux outils théoriques et numériques pour les SDNR. Ces outils doivent être à la fois suffisamment simples pour faciliter l'analyse authentique du système, les simulations et la conception des contrôles, mais aussi suffisamment sophistiqués pour capturer les propriétés qualitatives de la réalité. Même si les SDNR existent dans des domaines distincts, ils concernent les mêmes modèles mathématiques et par suite le même intérêt à les étudier.

On utilise souvent des équations différentielles avec des lois non-différentiables, des inclusions différentielles ou des inclusions différentielles à mesures dans le but d'étudier l'évolution des SDNR dans le temps. Les scientifiques sont intéressés par l'analyse, la stabilité des trajectoires et les simulations de ces modèles mathématiques pour les comparer avec les vraies données des phénomènes étudiés.

Malgré l'explosion récente de la théorie de la stabilité des systèmes dynamiques réguliers, la théorie de la stabilité des SDNR reste un champ de recherches relativement jeune où beaucoup de questions ouvertes subsistent. La terminologie utilisée pour les systèmes dynamiques réguliers peut parfois ne pas avoir de sens pour les SDNR. Les chercheurs ont donc des difficultés à faire face à certaines catégories spécifiques de SDNR, ce qui les conduit à développer de nouveaux outils analytiques et numériques pour l'étude de ces systèmes. Ce mémoire est consacré à l'étude des propriétés de stabilité des SDNR. Les applications considérées sont issues de la théorie des circuits électriques en électronique analogique ainsi que de la mécanique, mais elles pourraient être appliquées et étendues à d'autres domaines des sciences appliquées. Nous utilisons différents outils d'analyse non-lisse pour modéliser, analyser et simuler certains circuits simples RLC, convertisseurs DC-DC de Buck, les systèmes dynamiques Lagrangiens et de Lur'e. Une attention particulière sera donnée à l'étude de l'analyse de stabilité de ces systèmes au sens de Lyapunov.

0.2 Théorie de la stabilité: Un bref aperçu historique

Ce paragraphe est inspiré des travaux de Van De Wouw & Leine [76, 77, 78, 79, 118].
Dans la longue histoire de la théorie de la stabilité, il existe différents concepts de stabilité qui ont apporté une contribution remarquable à l'avancement de la théorie. Citons par exemple les contributions de Lagrange, Poisson et Lyapunov.... La définition exacte de la stabilité et de la théorie de l'équilibre général pour les systèmes non-linéaires a été mise en place en 1892 par le mathématicien et mécanicien A. M. Lyapunov (1857-1918). Cette théorie est très utilisée aujourd'hui avec des applications potentielles dans plusieurs domaines.

En 1644, E.Torricelli $(1608 - 1647)$ a déclaré son axiome (traduction en français moderne):

> *Deux poids, qui sont reliés entre eux ne peuvent pas commencer à se déplacer par eux-mêmes si leur centre commun de gravité ne descend pas.*

Nous devons comprendre "commence à se déplacer par eux-mêmes" ici comme une perturbation de l'équilibre et sans doute à ses préoccupations en matière de déclaration avec le concept de base de la stabilité sur l'énergie potentielle. S. Stevin $(1548 - 1620)$ et C. Huygens $(1629-1695)$ ont fait des recherches sur l'équilibre et la stabilité de corps flottants. Au XVIIIe siècle, Daniel Bernoulli $(1700 - 1782)$, Pierre Bouguer $(1698 - 1758)$, Leonhard Euler $(1707 - 1783)$ ont travaillé sur le déploiement et la stabilité des navires. Bernoulli a clairement défini la stabilité (qu'il appelait "fermeté") de l'équilibre. Euler a amélioré le travail de Bernoulli et d'abord utilisé le mot " stabilité" d'impliquer avec une infime perturbation de l'emplacement de l'équilibre. Il a également frayé un chemin dans la théorie de la stabilité élastique en statique avec Bernoulli. Le travail de Bouguer est indépendant de Euler. Il est soupçonné d'être la première personne qui avait introduit le terme "hauteur métacentrique", qui a été utilisé dans le déploiement et la stabilité des navires jusqu'à maintenant.

Mathématicien, mécanicien et astronome, J. L. Lagrange (1736 - 1813) a continué avec sa formulation de Torricelli axiome des systèmes dynamiques conservateur en utilisant le concept d'énergie potentielle. Lagrange est vraisemblablement le premier à avoir étudié la stabilité dans le sens moderne du terme. L'une de ses conclusions était qu'en absence d'une force extérieure, si le système est conservatif, un état correspondant à une énergie cinétique nulle et à une énergie potentielle minimale, est un point d'équilibre stable.
J. P. G. Lejeune Dirichlet (1805-1859) a ajouté à la conclusion de Lagrange que le minimum de l'énergie potentielle est une condition suffisante pour prouver la stabilité. La théorie est donc considérée comme le théorème de stabilité Lagrange-Dirichlet dans la littérature. En outre, des études en mécanique céleste du P. S. Laplace (1749-1827), S. D. Poisson (1781-1840), C. G. J. Jacobi (1804-1851), H. Poincaré (1854-1912)... ainsi que des recherches dans l'analyse de la stabilité des machines de J. C. Maxwell (1831 - 1879), E. J. Routh (1831 - 1907), I. I. Vishnegradsky (1893 - 1979), A. B. Stodola (1859 - 1942), A. Hurwitz (1859 - 1919)... ont grandement contribué à la théorie de la stabilité moderne.

Les premières définitions mathématiques exactes de la stabilité remontent au XIXe siècle et sont dues à des scientifiques russes. En 1882, Nikolai Egorovich Zhukovskii (1847-1921), qui est à l'origine de l'hydrodynamique et de l'aérodynamique moderne a présenté un nouveau concept sur la forte stabilité orbitale basée sur une reparamètrisation de la variable temps. Sa contribution a été largement éclipsée par les travaux de Henri Poincaré (1854-1912) ainsi que par le succès d'un autre savant russe, Alexandre Mikhaïlovitch Lyapunov (1857-1918). Le 12 Septembre 1892, Lyapunov a défendu sa célèbre thèse de doctorat "*Le problème général de la stabilité du mouvement*" à l'université de Moscou. Dans cette thèse, il a introduit les définitions de base sur la stabilité qui sont encore en usage aujourd'hui et ses résultats fondamentaux, ont permis d'analyser les équations différentielles ordinaires (dans la méthode de Lagrange, son travail a été limité à un système mécanique conservateur). Ses traités sur la classifiation des différents types de stabilité: la stabilité, la stabilité asymptotique locale, la stabilité asymptotique globale sont à la base des études sur les systèmes non-linéaires. Lyapunov a mis en lumière deux approches distinctes pour traiter le problème de la stabilité des systèmes linéaires et non linéaires. La premiere est connue comme la *première méthode de Lyapunov* ou *méthode indirecte* de Lyapunov. Cette méthode consiste à linéariser le système pour étudier la stabilité locale du système original.

La deuxième méthode de Lyapunov appelée aussi *méthode directe de Lyapunov*, qui a l'avantage d'étudier la stabilité par le seul examen de ł'energie globale du système considèré. Elle est basée sur la considération physique suivante: *si l'énergie totale d'un système est dissipée de manière continue, alors le système (linéaire ou ou non-linéaire) doit atteindre un point d'équilibre.*

La théorie élégante de la stabilité de Lyapunov est restée inconnue à l'Ouest jusqu'aux années 60. Cependant, la structure de la théorie est de nos jours bien connue et la théorie est un outil fondamental dans l'étude moderne des systèmes dynamiques et dans la théorie du contrôle.

0.3 Systèmes Dynamiques Non-Réguliers (SDNR)

L'absence de régularité pour la plupart des systèmes réels est inévitable en raison de leur complexité. En effet, les phénomènes non-réguliers jouent un rôle essentiel dans la mécanique et les mathématiques appliquées. Ils peuvent provenir des systèmes mécaniques avec impacts, contraintes unilatérales, ou des frottement de Coulomb. Ils apparaissent également dans les réseaux électriques avec des dispositifs de commutation tels que diodes, diacs, thyristors, transistors ... ainsi que dans la théorie du contrôle lorsque les conceptions de contrôle discontinues sont utilisés. Au cours des trois dernières décennies, les SDNR ont été largement étudiés montrant une complication notable des réponses dynamiques, même pour des effets simples.

Nombreux sont les problèmes en ingénierie qui engendrent des inéquations dans leur formulation mathématique et contiennent donc nécessairement une non-régularité intrinsèque. Cette non-régularité peut trouver son origine dans le terme discontinu du contrôle, ou dans l'environnement du système (comme les problèmes d'impacts par exemple), ou bien du problème de frottement sec de type Coulomb. Ces systèmes sont appelés systèmes dynamiques unilatéraux ou non-réguliers. Le formalisme des inéquations variationnelles d'évolution semble représenter une large classe de systèmes dynamiques unilatéraux. A cause de cette perte de régularité, les méthodes classiques connues pour les équations différentielles ordinaires ne peuvent être appliquées aux systèmes dynamiques unilatéraux. Par conséquent, une extension naturelle de ces résultats au cas non-régulier est nécessaire. Comme première approche, nous pouvons citer les travaux de Filippov [55, 56] pour les équations différentielles avec second membre discontinu. Récemment, de nouveaux outils ont été développés pour l'étude de la stabilité des inéquations variationnelles d'évolution [63], [35], [36], [33], [34] etc ....
Dans cette section, nous essayons de faire ressortir clairement la terminologie liée aux SDNR, et de donner une classification de ces derniers.

C'est un fait qu'il n'y a pas de classification unifiée des SDNR. Le terme "systèmes dynamiques non-réguliers" est utilisé vaguement dans la littérature en raison de convention, mais l'interprétation explicite fait ressortir que les propriétés de ces systèmes sont "non-régulières". C'est la raison pour laquelle diverses classifications des SDNR sont introduites par de nombreux auteurs. Il est normal de penser que le terme "non-régulier" est essentiellement lié à la régularité du membre de droite f dans les équations différentielles, mais cette analyse peut s'avérer plus compliquée à étudier. En s'appuyant sur les références [1, 2, 7, 77, 106], nous allons diviser les SDNR en 3 classes :

1. **Les systèmes continus non-réguliers:** décrits par des équations différentielles $\dot{\mathbf{x}}(t) = f(\mathbf{x}, t)$ où le terme de droite (également appelé champ de vecteurs) est lipschitzien mais non dérivable en $\mathbf{x}$ en certaines hyper-surfaces dans l'espace d'état. Un oscillateur mécanique avec un support élastique d'un côté et de l'excitation extérieure peut être considéré comme un exemple.

2. **Les systèmes discontinus ou les systèmes de Filippov:** décrit par des équations différentielles $\dot{\mathbf{x}}(t) = f(\mathbf{x}, t)$ où les champs de vecteurs sont discontinus. Les systèmes sont également appelés systèmes Filippov parce que leurs solutions doivent être com-

prises dans le sens de Filippov [55]. La dérivée $\dot{\mathbf{x}}(t)$ ne peut pas être définie partout à cause de la discontinuité du côté droit. Des exemples peuvent être trouvés dans les systèmes visco-élastiques et de frottement sec ou dans les modèles de convertisseurs de tension électroniques [28, 57]. Les systèmes dont l'état est continu en temps, mais décrit par un ensemble de lois non-régulieres (équations généralisées) entre l'etat $\mathbf{x}$ et les sorties ou les multiplicateurs de Lagrange λ. Les lois non-régulieres peuvent être des conditions de complémentarité, des inéquations variationnelles, des inclusions différentielles, ou des projections . . . A titre d'exemples nous pouvons citer les circuits électriques contenant des diodes idéales ou diodes Zener idéales, DIACs, Thyristors ou transistors.

3. **Les systèmes avec sauts dans l'état:** l'état évolue de manière discontinue dans le temps. Par exemple les problèmes d'impact ou les systèmes avec changement brusque de vitesse ou les machines vibro-impacts.

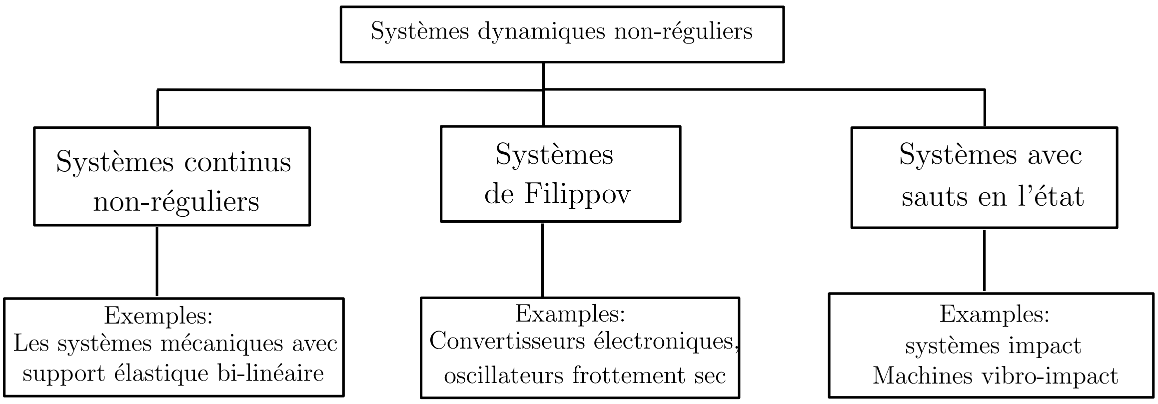

Figure 1: Classification de SDNR.

L'explosion de la recherche sur les SDNR dans les dernières décennies conduit à différents cadres mathématiques pour aborder de tels systèmes, tels que la dynamique unilatérale, les systèmes dynamiques, problèmes de complémentarité, systèmes réguliers par morceaux, les inéquations variationnelles d'évolution, systèmes hybrides, équations différentielles discountinues, inclusions différentielles à mesures [1, 2, 34, 77]. Chaque cadre possède son propre avantage qui le rend plus adapté à certains objectifs par rapport aux autres. Dans de nombreux cas, ces cadres mathématiques peuvent être équivalents les uns avec les autres [34].

0.4 Enquête dans la littérature de l'analyse non-lisse et de la stabilité dans les SDNR

Dans cette section, nous donnons un résumé des ouvrages fondamentaux, des articles et d'autres sources pertinentes sur la stabilité des systèmes dynamiques non-réguliers. Tout d'abord, nous proposons une synthèse des résultats fondamentaux en analyse non-lisse qui sont nécessaires à l'étude des SDNR. Après un examen des publications récentes sur la stabilité dans les systèmes non-réguliers, nous en présentons ici les principaux résultats.

0.4.1 Analyse non-lisse

L'analyse convexe est la branche de l'analyse non-lisse qui s'occupe essentiellement de l'étude des propriétés des ensembles et des fonctions convexes. Il est incontestable que l'analyse convexe joue un rôle important dans l'analyse non-lisse. Nous renvoyons le lecteur à quelques ouvrages de références qui concernent l'analyse convexe : tout d'abord le livre classique de T. Rockafellar [102], les ouvrages de T. Rockafellar et R. Wets [103], de J. J. Moreau [90, 91, 92], ainsi que les travaux de J.B. Hiriart–Urruty et C. Lemarèchal [68, 69]. Le livre [48] de F. Clarke ainsi que F. H Clarke, Ledyaev, Y. S., Stern, R. J., et Wolenski, P. R. [49] sont très utiles dans l'analyse non-lisse et ses applications. Des travaux importants sur les fonctions multivoques peuvent être trouvés dans les livres [19, 20, 21] par J. P. Aubin-Cellina, J. P. Aubin-I. Ekeland ainsi que J. P. Aubin et H. Frankowska.

Les inclusions différentielles sont une généralisation de la notion d'équation différentielle ordinaire, où le membre de droite est une fonction multivaluée. L'un des premiers mathématiciens ayant significativement contribué à cette théorie est Filippov [55, 56] avec ses extensions pour l'équation différentielle discontinue. Les inclusions différentielles sont également étudiées dans [19] par J.P. Aubin et A. Cellina, le livre [49] par Clarke *et al*, dans [51] par Deimling ainsi que dans [111] par Smirnov. Les inclusions différentielles à mesures sont plus générales que les inclusions différentielles de type Filippov. Elle permettent de décrire les systèmes avec des sauts en l'état, mis en place par Moreau [90, 91] pour l'étude des problèmes de rafles. Nous pouvons aussi citer l'ouvrage de M. Monteiro Marques [84] ainsi que D. Stewart [113].

0.4.2 Stabilité des SDNR

Le nombre de publications sur la théorie de la stabilité pour les systèmes non-réguliers est énorme et croit de manière intensive avec différents formalismes. Nous pouvons par exemple citer [2, 42] pour les systèmes de complémentarité, [66, 82] pour des systèmes de commutation, [72] pour les systèmes affines par morceaux, [41, 100] pour les systèmes hybrides, [22, 75] pour systèmes dynamiques impulsifs ...
La stabilité des jeux d'équilibre et de propriété dichotomie (c.-à-d. la convergence des solutions ou absence de limites) des inclusions différentielles est étudiée par Yakubovich, Leonov, Gelig in [119]. En effet, ils ont montré comment construire une fonction de Lyapunov pour garantir la stabilité de l'équilibre. Ce travail mène à la recherche de solutions pour des inégalités matricielles linéaires LMI.

La stabilité des ensembles de points d'équilibres des inclusions différentielles décrivant les systèmes avec frottement en mécanique et en électronique ont été étudiés par Adly, Attouch, Cabot, Goeleven, Brogliato, Motreanu [5, 6, 8, 9, 10, 11, 33, 35, 36, 37, 62, 63] et Van De Wouw, Leine [77, 78, 79, 118]. La solution existe toujours et est unique quand l'opérateur est maximal monotone . Les références [5, 6, 8, 10, 35] donnent un théorème de Lyapunov pour la stabilité et l'attractivité d'inclusions différentielles du premier et second second ordre. Les inclusions différentielles du second ordre sont discutées dans [9] par l'étude des systèmes dynamiques de type Lagrange avec un contrôleur de jeu-évalués. Dans cet article, le problème d'existence, d'analyse de stabilité, y compris la stabilité en temps fini sont étudiés en utilisant une fonction de Lyapounov régulière choisie en fonction des propriétés particulières des systèmes.

Dans [33], B. Brogliato utilise des inclusions différentielles à mesure pour décrire les systèmes mécaniques avec un impact sans frottement. Une généralisation du théorème de stabilité de type Lagrange-Dirichlet a été étudiée. Des travaux plus généraux peuvent être trouvés dans [46, 47] par Chareyron et Wieber. En outre, le principe d'invariance de Lasalle est étendu [35] par B. Brogliato et D. Goeleven aux inéquations variationnelles d'évolution et aux systèmes avec contraintes unilatérales (en particulier, les systèmes mécaniques avec frottement et contact unilatéral).

Dans [61], D. Goeleven et B. Brogliato donnent des résultats d'instabilité en dimension finie pour des inéquations variationnelles d'évolution. Des résultats d'instabilité pour une classe d'inéquations variationnelles paraboliques dans l'espace de Hilbert est donnée par Quittner [101]. Il est connu que dans de nombreux cas, trouver une fonction de Lyapunov régulière est impossible. Par conséquent, il est naturel de développer la théorie de la stabilité aux fonctions de Lyapunov non-lisses, voir par exemple les travaux de S. Adly, A. Hantoute et M. Théra [12, 13] ainsi que les références suivantes [23, 24].

0.5 Objectif et aperçu de la thèse

L'objectif principal de cette thèse est de proposer une formulation pour l'étude et l'analyse de stabilité des systèmes dynamiques non-réguliers avec une attention particulière aux applications issues des circuits électriques et des systèmes mécaniques avec frottement sec. Les outils mathématiques que nous utiliserons sont issus de l'analyse non-lisse et de la théorie de stabilité au sens de Lyapounov. Dans le détail, nous utilisons un formalisme pour modéliser la complémentarité des systèmes de commutation simples et des inclusions différentielles pour modéliser un convertisseur DC-DC de type Buck, les systèmes dynamiques Lagrangien ainsi que les systèmes de Lur'e. Pour chaque modèle, nous nous intéressons à l'existence d'une solution, des propriétés de stabilité des trajectoires, de la stabilisation en temps fini ou de mettre une force sur la commande pour obtenir la stabilité en temps fini. Nous proposons aussi quelques méthodes numériques pour simuler ces systèmes. Il est à noter que les méthodes utilisées dans ce manuscrit peuvent être appliquées pour l'analyse de systèmes dynamiques non-réguliers issus d'autres domaines tels que l'économie, la finance ou la biologie.

Le manuscrit est divisé en 6 chapitres qui sont étroitement liés. Dans le chapitre 1, nous donnons une courte introduction et une motivation pour l'étude de la stabilité des systèmes dynamiques non-réguliers et nous proposons quelques références de base importantes pour la suite de notre travail. Le chapitre 2 contient les résultats fondamentaux, utile pour la suite de notre travail, sur l'analyse non-lisse , la théorie de Filippov et la théorie de stabilité au sens de Lyapunov. Le Chapitre 3 est consacré à l'analyse des circuits électriques.

Tout d'abord, nous rappelons les caractéristiques ampères-volts de certains composants électriques, y compris certains périphériques non réguliers comme les diodes, les thyristors, diacs ou les transistors. Puis, quelques circuits simples de type RLC avec diodes sont analysés à l'aide de la formulation de la complémentarité. Dans la partie en fin de chapitre, un convertisseur DC-DC de Buck est modélisé par une inclusion différentielle. Nous montrons que le problème est bien posé, puis nous donnons des résultats de stabilité. Le chapitre 4 porte sur les systèmes dynamiques Lagrangien soumis à une force de perturbation gouvernés par une inclusion différentielle du second ordre. Dans ce chapitre, les propriétés d'existence de solutions et de stabilité des trajectoires sont analysées avec soin. Nous montrons que sous la condition de frottement sec, les trajectoires convergent vers l'équilibre en un temps fini, puis nous donnons une estimation de ce dernier. Dans le chapitre 5, l'analyse de la stabilité des systèmes généraux de type Lur'e est étudiée. Nous généralisons des résultats récents de B. Brogliato et D. Goeleven [36] au cas d'une inclusion différentielle non-monotone. Les résultats théoriques sont étayés par quelques exemples dans les circuits électriques, puis nous présentons quelques simulations numériques de ces derniers. Le chapitre 6 présente une conclusion puis quelques perspectives de recherches.

Chapitre 3: Résultats d'existence et de stabilité pour les circuits électriques

Un circuit électrique est un réseau constitué de plusieurs composants électroniques et d'une source de courant ou de tension. Il peut contenir des éléments électriques dits réguliers tels que des résistances, inductances, condensateurs, et des éléments dits non-réguliers tels que les diodes, les DIAC, les thyristors, ou les transistors ainsi que des sources de tension, sources de courant, ou interrupteurs. Bien que l'analyse des circuits linéaires soit beaucoup plus facile, celle du cas non-linéaire contient encore des problèmes ouverts. Par conséquent, l'étude des circuits électriques par des méthologies de type SDNR a un rôle important dans la pratique et peut également être appliquée dans d'autres domaines. En effet, les caractéristiques des diodes en électronique; friction et impact de la mécanique, les lois de commande de commutation dans la gestion du trafic aérien, les modèles économiques des marchés... peuvent être modélisées par des lois non-lisses avec des structures similaires. Nous étudions les circuits électriques de ce chapitre par le biais de problème de complémentarité et d'inclusions différentielles. Tout d'abord, nous présentons les caractéristiques ampères-volts de certains dipôles électriques, y compris certains composants non-lisses comme les diodes, redresseurs silicium, diacs, trias ou transistors ... Certains circuits simples RLCD avec diodes peuvent-être analysés en utilisant le formalisme de complémentarité. Enfin, certains convertisseurs DC-DC de type Buck sont modélisés par l'inclusion différentielle de la forme:

$$LCv''(t) + \frac{L}{R}v'(t) + v(t) \in V_{in}u(t, v(t)) \tag{2}$$

où $u(t, v)$ est définie par:

$$u(t, v) = \frac{1}{2}\Big(\text{Sign}(V_r(t) - v) + 1\Big)$$

avec

$$V_r(t) = V_{ref} + \frac{1}{a} v_r(t).$$

La fonction Sign : $\mathbb{R} \rightrightarrows \mathbb{R}$, $x \mapsto \text{Sign}(x)$ est la fonction multivoque définie par:

$$\text{Sign}(x) = \begin{cases} -1 & \text{if} \quad x < 0, \\ [-1, +1] & \text{if} \quad x = 0, \\ +1 & \text{if} \quad x > 0. \end{cases}$$

Des résultats d'existence et de stabilité sont données dans cette section. L'approche de Filippov a été largement appliquée dans les systèmes mécaniques et elle peut être également utilisée dans les circuits électroniques de puissance. Ce chapitre est un travail en collaboration avec Prof. S. Adly et Prof. D. Goeleven.

Chapitre 4: Systèmes dynamiques Lagrangien avec contrôleurs multivalués

Il est connu que les fondements de la plupart de la mécanique classique ont été introduits par Isaac Newton dans son monographe *Philosophiæ Naturalis Principia Mathematica*, publié en 1687. Trois de ses fameuses lois du mouvement ont eu un grand impact sur la vision scientifique du monde physique pendant plus de trois siècles. Dans les années 1750, le mathématicien suisse Leonhard Euler et le mathématicien français Joseph Louis Lagrange ont développé un outil pour analyser de nombreux problèmes réels de minimisation de longueur d'arc, tel que le problème brachistochrone ou bien encore les problèmes géodésiques en résolvant une équation différentielle. Cette équation est appelée équation d'Euler-Lagrange et joue un rôle fondamental dans le calcul des variations. L'équation est équivalente à la deuxième loi du mouvement de Newton en mécanique classique, mais il est plus pratique de l'avoir dans n'importe quel système de coordonnées généralisées. En effet, en substituant l'expression du lagrangien dans l'équation de Lagrange, nous obtenons les équations du mouvement du système. Notez que le lagrangien du système n'est pas unique, même si la résolution de tout lagrangien équivalent donne les mêmes équations du mouvement [64]. La méthode est largement utilisée en mécanique classique, en automatique et en théorie du contrôle pour résoudre des problèmes d'optimisation ainsi que l'analyse des propriétés désirées de trajectoires des systèmes. Il joue un rôle important non seulement en raison de vastes applications, mais aussi pour faire avancer la compréhension approfondie des phénomènes réels.

Afin d'établir l'équation de Lagrange pour le mouvement d'un objet, il est nécessaire de trouver *l'énergie potentielle* $\mathcal{V}(q)$(l'énergie stockée par exemple quand un ressort est comprimé ou quand un objet est accroché à une hauteur) et le *l'énergie cinétique* $\mathcal{T}(q, \dot{q})$ (découle de la proposition) avec des coordonnées généralisées $q \in \mathbb{R}^n$. Ensuite, le lagrangien $\mathcal{L}(q, \dot{q})$ est définie par la différence entre l'énergie cinétique et l'énergie potentielle:

$$\mathcal{L}(q, \dot{q}) = \mathcal{T}(q, \dot{q}) - \mathcal{V}(q),$$

où l'énergie cinétique est habituellement donnée par la forme quadratique:

$$\mathcal{T}(q,\dot{q}) = \frac{1}{2}\langle M(q)\dot{q},\dot{q}\rangle,$$

est l'énergie potentielle. La matrice $M(q) \in \mathbb{R}^{n\times n}$ est appelée la matrice des masses ou l'inertie, qui est supposée définie positive et satisfait certaines conditions supplémentaires que nous allons vérifier plus tard. En utilisant les coordonnées généralisées $q \in \mathbb{R}^n$ et l'entrée de commande $u \in \mathbb{R}^n$, l'équation de Lagrange peut s'écrire sous la forme:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = u. \tag{3}$$

En utilisant le premier type de symboles de Christoffel [107, 112], on peut réécrire (3) sous la forme:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla\mathcal{V} = u, \tag{4}$$

où $C(q,\dot{q})$ est appelée la matrice de Coriolis (et / ou moments) : matrice qui comprend les termes d'effets centrifuges et de Coriolis. Le terme $C(q,\dot{q})\dot{q}$ est la force centrifuge et la force de Coriolis où $\nabla\mathcal{V}$ contient les forces gravitationnelles. On peut vérifier facilement que:

$$C(q,\dot{q})\dot{q} = \frac{d}{dt}\big(M(q(t))\dot{q} - \frac{1}{2}\left(\frac{\partial \mathcal{T}(q,\dot{q})}{\partial q}\right). \tag{5}$$

Le terme $\frac{d}{dt}\big(M(q(t)\big) - 2C(q,\dot{q})$ est une matrice antisymétrique [112], où

$$\frac{d}{dt}\big(M(q(t)\big) = \sum_{i=1}^{n} \dot{q}_i \frac{\partial M}{\partial q_i}.$$

En effet, la matrice $C(q,\dot{q})$ est donnée par:

$$C_{jk}(q,\dot{q}) = \sum_{i=1}^{n} C_{ijk}(q)\dot{q}_i, \tag{6}$$

où C_{ijk} est définie par les symboles de Christoffel de première espèce:

$$C_{ijk} = \frac{1}{2}\left(\frac{\partial M_{jk}}{\partial q_i} + \frac{\partial M_{ji}}{\partial q_k} - \frac{\partial M_{ik}}{\partial q_j}\right). \tag{7}$$

De (6), nous pouvons écrire la matrice $C(q,\dot{q})$ sous la forme :

$$C(q,\dot{q}) = \sum_{i=1}^{n} \dot{q}_i C_i(q), \tag{8}$$

où $C_i(q)$ possède les entrées $C_{ijk}(q)'s$ et satisfait l'égalité $C_i + C_i^T = \frac{\partial M}{\partial q_i}$, $i = 1,2,\ldots,n$. Par conséquent, nous pouvons calculer la force centrifuge et de Coriolis à travers la matrice d'inertie et on obtient la relation :

$$\frac{d}{dt}\big(M(q(t)\big) = C(q,\dot{q}) + C(q,\dot{q})^T. \tag{9}$$

Dans ce chapitre, nous étudions le système (4) qui est soumis à une force de perturbation $F(\cdot, q, \dot{q})$ et le terme de contrôle u est la force de la forme $u \in \partial\Phi(\dot{q})$ (par exemple, la force de frottement de type Coulomb) où Φ est une fonction scalaire convexe pour stabiliser le système. Par conséquent, nous considérons une classe de systèmes dynamiques non linéaires de type Lagrange avec un contrôleur de valeurs multivaluées de la forme :

$$M(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t) + \nabla\mathcal{V}(q(t)) + F(t, q(t), \dot{q}(t)) \in -\partial\Phi(\dot{q}(t)) \tag{10}$$

pour *p.p.* $t \geq t_0$, où $t_0 \in \mathbb{R}$ est fixé, $\Phi : \mathbb{R}^n \to \mathbb{R}$ est une fonction convexe, $\mathcal{V} \in \mathcal{C}^1(\mathbb{R}^n; \mathbb{R})$ avec son gradient $\nabla\mathcal{V}(\cdot)$, les matrices $M(q), C(q, \dot{q}) \in \mathbb{R}^{n\times n}$, et $\partial\Phi(\cdot)$ représente le sous-différentiel convexe de $\Phi(\cdot)$. Motivées par des applications de la théorie du contrôle, nous considérons dans ce travail le cas où la fonction multivaluée dans (10) dépend de la vitesse uniquement. D'autres travaux [98, 108] ont étudié le cas où la partie multivoque est un cône normal associé à un ensemble convexe i.e. $\partial\Phi = N_C$, qui dépend de la position uniquement. On obtient ainsi différents types de dynamique avec contraintes unilatérales, et discontinuités de la vitesse. La force de perturbation fonction $F(t, q(t), \dot{q}(t))$ est généralement bornée par une constante. L'entrée de commande $\partial\Phi(\dot{q})$ et la force de gravitation $\nabla\mathcal{V}$ sont appliquées pour stabiliser le système (en temps fini) à un point fixe. L'avantage de ces contrôleurs est qu'ils sont intrinsèquement robuste parce que la connaissance *a priori* des paramètres du système (comme les paramètres inertiels) n'est pas nécessaire pour la stabilisation, et une borne supérieure de la perturbation est suffisante pour le rejeter. En outre, les trajectoires du système en boucle fermée atteignent le point d'équilibre en temps fini. Le contrôleur robuste discontinu est un domaine de recherche important dans les systèmes et contrôle [16, 17, 25, 58, 74, 95, 115, 116, 120], où les dites entrées de mode de glissement sont analysées. Les contrôleurs robustes garantissant une stabilité en temps fini sont appliquées. De nombreuses autres applications, aux systèmes mécaniques et électro-mécaniques se trouvent dans [54, 67, 71, 110, 117, 121]. Les propriétés de convergence en temps fini sont également étudiées dans la littérature mathématique et de contrôle [8, 40, 55, 88, 89], ainsi que des propriétés de stabilité et d'invariance des systèmes non lisse et des inclusions différentielles [19, 23, 35, 36, 63, 77, 86, 95, 105]. Beaucoup de ces travaux sont basés sur les inclusions dites différentielles de type Filippov [55]. Dans d'autres travaux, la maximale monotonie de l'opérateur engendré par l'inclusion est une propriété centrale, comme dans [3, 10, 32, 33, 35, 36]. Dans ce chapitre, la dynamique en boucle fermée dans (10) est analysée. L'existence et l'unicité des solutions est d'abord soigneusement étudiées, et les propriétés de stabilité et d'attractivité de l'ensemble des points d'équilibre sont examinées. Les outils sont ceux de l'analyse convexe, qui sont combinés avec les propriétés des systèmes dynamiques de type Lagrange. Par rapport à certains travaux antérieurs portant sur une dynamique du second ordre similaire [10], dans ce chapitre, nous considérons le cas où la matrice des masses $M(q)$ dépend de la position q. Comme nous le verrons, ce n'est pas une tâche facile, car cela peut détruire la monotonie de l'opérateur qui apparaît dans la formulation du premier ordre de (10), *i.e.* $z = (z_1, z_2) \mapsto M^{-1}(z_1)\partial\Phi(z_2)$. En d'autres termes, le changement de variables utilisé dans [3, 10, 33, 35, 36] pour reformuler le problème d'une inclusion différentielle du premier ordre engendrée par un opérateur maximal monotone, et qui utilise les règles de calcul du sous-différentiels d'analyse convexe [114, Theorem 4.2.1] ne fonctionne plus pour les matrices de masse non-triviales (dépendant de l'état q). L'unicité de la solution ne peut-être prouvée pour de tels systèmes. Nous proposons une analyse de stabilité pour des systèmes à solutions multiples.

Ce chapitre a fait l'objet d'un article paru dans SIAM Journal on Control and Optimasation en collaboration avec Prof. S. Adly et Prof. B. Brogliato.

Chapitre 5: Systèmes dynamiques de Lur'e

Les systèmes dynamiques de Lur'e multivoques ont été largement étudiées dans les communautés du contrôle et des mathématiques appliquées (voir [81]). Les systèmes généraux de Lur'e sont des systèmes qui ont une interconnexion avec rétroaction négative d'une équation différentielle ordinaire $\dot{x}(t) = f(x(t), p(t))$ où p est l'une des deux variables d'écart, avec le second $q = g(x,p)$ et satisfaisant la condition d'inclusion $p \in \Phi(q,t)$. Notons que d'autres modèles mathématiques utilisés pour étudier les systèmes dynamiques non-réguliers (systèmes relais, les inéquations variationnelles, les systèmes dynamiques projetés, la complémentarité des systèmes...) peuvent être également reformulés en des systèmes de Lur'e (voir par exemple [34, 52, 65]).

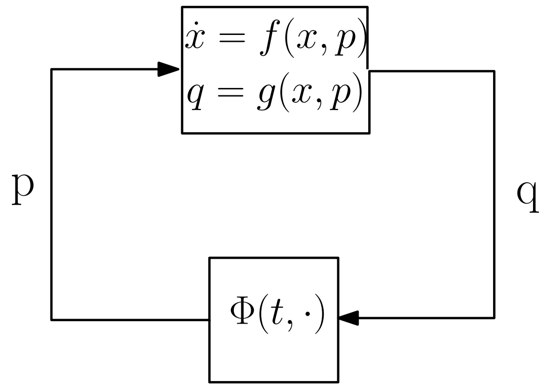

Figure 2: Systèmes Lur'e.

Nous nous sommes intéressés par les systèmes Lur'e qui sont (peuvent-être non linéaire) invariables dans le temps contenant une partie multivoque. Habituellement, la fonction g est supposée linéaire i.e. de la forme: $g(x,p) = Cx + Dp$, où C et D sont deux matrices. Le cas $D = 0$ apparaît dans de nombreuses applications en électronique. Le cas $D \neq 0$ est plus général mais crée des difficultés quand on veut étudier la valeur de l'opérateur $(-D + \Phi^{-1})^{-1}(\mathcal{C} \circ \cdot)$. Dans [36], Brogliato et Goeleven ont surmonté ces obstacles en supposant que la partie multivoque est représentée par le sous-différentiel d'une fontion convexe, sci et propre. En utilisant une transformation adéquate, les auteurs ont montré que le système de Lur'e peut-être étudié dans le cadre d'inéquation variationnelle d'évolution. Ceci a été établi grâce aux bonnes propriétés de la fonction conjuguée de Fenchel et du sous-différentiel convexe. Cependant, le cas de $D = 0$ est toujours intéressant car il comprend des applications dans la pratique tels que les circuits électriques avec certains périphériques où leurs caractéristiques courant-tension ne sont pas monotones. Il y a certainement beaucoup de résultats dans la littérature pour le cas où la matrice $D = 0$, mais très peu de résultats existent pour le cas où la partie multivoque est non-monotone [35, 37]. C'est le but de ce travail, que l'on espère comblera certaines lacunes dans la littérature pour l'étude des systèmes de Lur'e non-monotones.

Dans ce chapitre, nous reformulons une classe de systèmes de Lur'e dont la forme différentielle du premier ordre est engendrée par un opérateur non-monotone à valeurs convexes et compactes. Nous établissons un résultat d'existence de solutions, ensuite nous étudions l'unicité de la solution en étudiant le cas où l'opérateur est localement hypomonotone. Ensuite, nous donnons une analyse de la stabilité et d'extension du principe d'invariance de LaSalle pour de tels systèmes. Enfin, pour valider les résultats théoriques, nous proposons quelques simulations numériques pour des exemples issus de l'électronique.
Ce chapitre est un travail en collaboration avec Prof. S. Adly et paru dans le Journal Applicable Analysis.

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Le Ba Khiet

"We hold these truths to be self-evident, that all men are created equal,

that they are endowed by their Creator with certain unalienable Rights,

that among these are Life, Liberty and the pursuit of Happiness."

— United States Declaration of Independence (July 4, 1776) —

Contents

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Chapter 1
Introduction

In this chapter, we give a motivation for the study of stability of non-smooth dynamical systems with non-smooth analysis tools. Subsequently, a brief historical note on stability theory and an introduction about non-smooth dynamical systems are given. Then we present a short literature survey about the concerned topics. The last sections are about the objective, scope and outline of the thesis. This chapter is inspired by the thorough work of Van De Wouw, Leine [76, 77, 78, 79, 118].

1.1 Motivation

Stability theory is a very old subject in mathematics whose objective is to draw conclusion about the behavior of a system without finding its solution trajectories. Stability of a dynamical system is a fundamental demand for its practical use. In many cases, it is very dangerous if the system is unstable. The important role of 'stability' in practice has motivated many scientists do research on it and therefore the stability theory has been considerably developed. Even though the theory must obviously be originated in mechanics but nowadays stability problems are essential not only in mechanics but also in physics, economics, biology, electrical engineering . . . with abundant applications. The theory is widely studied and has an indisputably crucial position in many branches of science.

On the other hand, engineers and scientists are frequently confronted with increasingly complex models which cannot be described by the usual classical ways. This requirement leads to the development of non-smooth dynamical system (NSDS) and non-smooth analysis tools. Especially in recent years, more and more scientists show their enthusiasm for this young, attractive area. In engineering, non-smooth dynamical systems exist everywhere such as stick-slip oscillation with dry friction, occurrence of impact . . . They are also present in electrical circuit, control systems as well as in various models from economy, biology . . . One usually uses smooth models to describe NSDS. Nevertheless, such smooth models cannot provide suitable predictions of the real systems. Therefore, non-smooth models need to be better understood in order to have an adequate and real predication. It is also the reason why scientists around the world always urge the development of new analytical and numerical tools for NSDS. These tools should be not only simple enough to support the authentic analysis, simulations and control design but also sophisticated enough to capture the qualitative properties of the real systems. Even though NSDS exist in different fields, they have quite same mathematical models and interests to study. One often uses differential equations with non-smooth laws, differential inclusions or measure differential inclusions to display their time evolution. People are interested in the well-posedness, stability analysis of trajectories and simulations of these mathematical models in comparison with the real ones in practice.

In contrast to the explosive development of stability theory for smooth dynamical systems, the stability properties of NSDS generally remain open. It's not even easy to define stability, attractivity... for such systems. Furthermore, terminology in smooth dynamical systems does not make sense in NSDS occasionally. Because of these difficulties, researchers can only deal with some specific classes of NSDS and always urge the development of new analytical and numerical tools for such systems. This monograph is dedicated to the study of stability properties of NSDS with applications particularly in electronics and mechanics, and may be extended and applied for other domains. We will use various non-smooth analysis tools to modelize, analyze and simulate for some simple RLC circuits, a DC-DC buck converter, Lagrange and Lur'e dynamical systems with a emphasis in the stability analysis in the sense of Lyapunov.

1.2 Stability Theory: A Brief Historical Overview

In the long history of stability theory, there exist various concepts of stability named after scientists who remarkably contributed to the advancement of the theory, e.g., Lagrange, Poisson and Lyapunov stability... The exact definition in the modern sense of stability and general stability theory for nonlinear systems were set up in 1892 by the mathematician and engineer A. M. Lyapunov (1857-1918) and are world - wide accepted today.

Look back in history, in 1644, E. Torricelli (1608 – 1647) stated his axiom (in modern English translation)

> *Two weights which are linked together cannot start moving by themselves if their common center of gravity does not descend.*

We may understand "start moving by themselves" here as a perturbation from an equilibrium and undoubtably his statement concerns the basis of the stability concept of potential energy. S. Stevin (1548 – 1620) and C. Huygens (1629 – 1695) did research on the equilibrium and stability issues of floating bodies, although Stevin's results are not generally true whereas Huygens' work still bases on geometrical methods. In the 18th century, Daniel Bernoulli (1700 – 1782), Pierre Bouguer (1698 – 1758), Leonhard Euler (1707 – 1783) studied about the roll-stability of ships. Bernoulli clearly defined the stability (which he called "firmness") of equilibrium, as well as discriminated between stable and unstable equilibriums. Euler improved Bernoulli's work and firstly used the word "stability" to involve with an infinitely small disturbance from the equilibrium location. He also pioneered in the theory of elastic stability in statics together with Bernoulli. Bouguer's work is independent from Euler's. He is believed to be the first person who introduces the term "metacentric height", which has been used in roll-stability of ships up to now.

Italian-born mathematician and astronomer J. L. Lagrange (1736 – 1813) continued with his formulation of the Torricelli axiom for conservative dynamical system using the potential energy concept. He may be the first person to study stability in the modern sense. One of his important conclusion was that, with the absence of external force, an equilibrium of a conservative mechanical system is stable if it corresponds to zero kinetic energy and minimum potential energy. J. P. G Lejeune Dirichlet (1805 − 1859) added to Lagrange's conclusion that minimum of the potential energy is sufficient to prove stability. The theory is therefore referred to as the Lagrange-Dirichlet stability theorem in the literature. In addition, studies in celestial mechanics by P. S. Laplace (1749 − 1827), S. D. Poisson (1781−1840), C. G. J. Jacobi (1804−1851), H. Poincaré (1854–1912)... as well as researches in machines stability analysis of J. C. Maxwell (1831−1879) , E. J. Routh (1831−1907), I. I. Vishnegradsky (1893−1979), A. B. Stodola (1859−1942), A. Hurwitz (1859−1919)... have vastly contributed to the modern stability theory.

The exact mathematical definitions of stability first came from Russian scientists in the nineteenth century. In 1882, N. E. Zhukovskii presented a new concept about the strong orbital stability based on a time variable reparametrization. His contribution did not receive many attentions since it partly agrees with the Poincaré stability and was shadowed by the great success of the Russian scientist Alexander Mikhailovitch Lyapunov (1857–1918). Lyapunov defended his famous doctoral thesis *The general problem of the stability of motion* in Moscow university on September 12, 1892. In this dissertation, he not only introduced the basic stability definitions that are in use nowadays but also proved many of crucial theorems, allowed us to analyze arbitrary differential equations (in Lagrange's method, his work was restricted to conservative mechanical system). His theory deals with various types of stability classifications: stability, local asymptotic stability, global asymptotic stability which are all important for nonlinear systems. Lyapunov showed two distinct approaches to deal with the stability problem for both linear and nonlinear systems. The first one is known as Lyapunov's first method or Lyapunov's indirect method, which depends on finding approximate solution through linearization. In the Lyapunov's second method or Lyapunov's direct method, there is a big advantage that no such knowledge is necessary. It makes one more convenient to handle the case of nonlinear systems. However, his elegant theory was largely unknown to the West until 1960's; therefore all achievements in Lyapunov stability theory until that time are only in Russia. Today, the substructures of the theory are well - found and the theory is an essential tool in both applied mathematics and engineering.

1.3 Non-smooth Dynamical Systems (NSDS)

Non-smoothness of most real systems is unavoidable because of their complexity. Indeed, non-smooth phenomena play an essential role in mechanics and applied mathematics. They may come from the mechanical systems with impacts, unilateral constraints, or Coulomb friction. They also appear in electrical networks with switching devices like diodes, diacs... as well as in control theory when discontinuous control designs are used. In the last three decades, NSDS have been vastly studied showing a noticeable complication of dynamical responses even for a simple impact or circuit. Such systems are not easy to deal with because of non-differentiable or even non-discontinuous right-hand side and sometimes having jumps in the state. However, thanks to the rapid development of the NSDS theory, we can now better understand those complexities. In this section, we try to make clear the terminology related to NSDS, and give a classification for NSDS.

It is fact that there is no a united classification for NSDS. The term "non-smooth dynamical systems" is used vaguely in the literature due to convention but without explicit interpretation that which properties of the systems are "non-smooth". This is the reason why various classifications for NSDS are introduced by many authors. It is normal to think that the term "non-smooth" comes from the right-hand side of differential equations but it may be much more complicated. With a reference from [1, 2, 7, 77, 106] we divide NSDS into 3 classes as following:

1. Non-smooth continuous systems: described by differential equations $\dot{\mathbf{x}}(t) = f(\mathbf{x}, t)$ where the right-hand side (also called a vector field) is Lipschitz continuous but not differentiable w.r.t to $\mathbf{x}$ at certain hyper-surfaces in the state-space. An example of non-smooth continuous systems described by a differential equation is given by a mechanical oscillator with a one-sided elastic support and external excitation.

2. Non-smooth discontinuous systems or Filippov systems: described by differential equations $\dot{\mathbf{x}}(t) = f(\mathbf{x}, t)$ where the vector fields are discontinuous on certain hyper - surfaces of the state - space. These systems are also called the Filippov systems because their solutions have to be understood in the sense of Filippov [55]. Even though $\dot{\mathbf{x}}(t)$ may not be defined everywhere because of the discontinuity in right - hand side, the state $\mathbf{x}(t)$ is still time - continuous. Examples can be found in systems with visco-elastic supports and dry friction or in the models of power electronic voltage converters [28, 57]. Systems with time-continuous state but described by a set of non-smooth laws (generalized equations) between the state $\mathbf{x}$ and outputs or Lagrange multipliers λ also belongs to this class. Non-smooth laws can be complementary conditions, differential inclusions, projections...For examples, models of electrical circuits with ideal diodes or ideal Zener diodes can be considered.

3. Systems with jumps in the state: the state evolves discontinuously in time and is not defined on such discontinuity time - instances. Examples are impacting systems with sudden change of velocity or vibro-impacting machines.

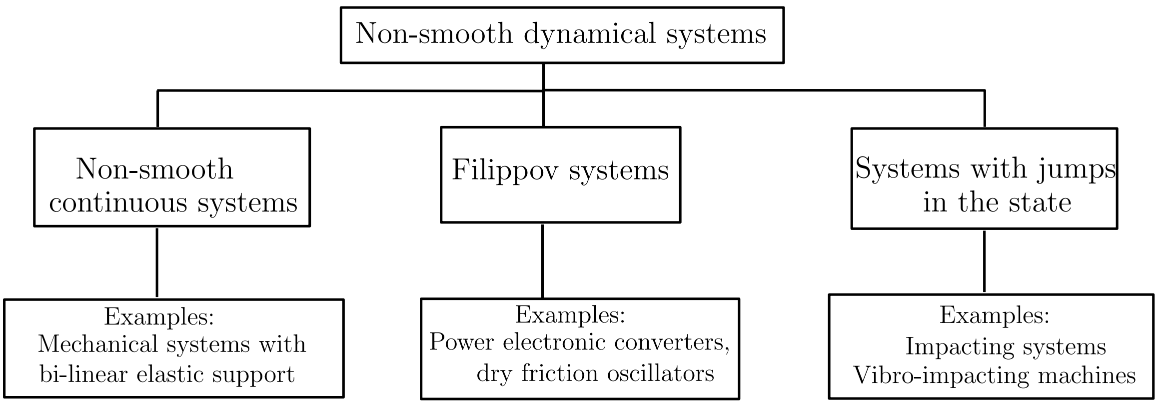


Figure 1.1: NSDS classification.

The explosion of research on NSDS in recent decades leads to different mathematical frameworks for investigating such systems, such as unilateral dynamics, dynamical complementarity systems, differential inclusions, piecewise smooth systems, evolution variational inequalities, differential variational inequalities, time varying systems, switched systems, hybrid systems, impulsive differential equations, measure differential inclusions [1, 2, 34, 77] ... Each framework has its own advantage which make it more suitable for some objectives than the others. In many cases, these mathematical frameworks can be proved to be equivalent with one or another [34].

1.4 Literature Survey of Non-smooth Analysis and Stability in NSDS

In this section, a summary of scholarly books, articles and other sources relevant to stability of non - smooth dynamical systems is given to help readers for background knowledge or for further references. Firstly, documents for non-smooth analysis including differential inclusions and measure differential inclusions are suggested which are necessary for studying of NSDS. After a review for them, recent publications on stability in non-smooth systems are presented.

1.4.1 Non-smooth Analysis

It is undoubted that convex analysis plays an important role in non-smooth analysis. For reference on convex analysis, one can find [102] by Rockafellar (classical book), [90, 91, 92] by J. J. Moreau, [103] by Rockafellar and Wets, as well as the work of Hiriart–Urruty and Lemarèchal [68, 69] . The book [48] by Clarke and the book [49] by Clarke and his coworkers are the fundamental ones for the theory of generalized gradients. Extensive works on set-valued applications can be found in the books [19, 20, 21] by J. P. Aubin-A. Cellina, J. P. Aubin-I. Ekeland and J. P. Aubin-H. Frankowska.

Differential inclusion is a generalization of the concept of ordinary differential equation, where the right-hand side is a set-valued function instead of single-valued map. One of the first mathematician developing the theory is Filippov [55, 56] with his extensions to the discontinuous differential equation. Differential inclusion is also introduced in [19] by Aubin and Cellina, the book [49] by Clarke *et al*, the book [51] by Deimling and [111] by Smirnov. Measure differential inclusions is more general than differential inclusions, allow us to describe the systems with jumps in state, introduced by Moreau [90, 91]. More references can be seen in Monteiro Marques [84], Stewart [113].

1.4.2 Stability in NSDS

The number of publications on stability theory for non-smooth systems is enormous and intensively increasing with various formalisms. We give a small reference, such as [2, 42] for complementarity systems, [66, 82] for switching systems, [72] for piece-wise affine systems, [41, 100] for hybrid systems, [22, 75] for impulsive dynamical systems... At the beginning of literature, most of the results are about the isolated equilibria.

Stability of equilibrium sets and dichotomy property (i.e. either convergence or unboundedness of solutions) in differential inclusions is studied by Yakubovich, Leonov, Gelig in [119]. In addition, they showed how to construct Lyapunov function guarantying global stability. This work leads to algebraically find a solution of matrix inequalities.

Stability of equilibrium sets of differential inclusions describing mechanical systems with friction and electronic systems is also studied by Adly, Attouch, Cabot, Goeleven, Brogliato, Motreanu [5, 6, 8, 9, 10, 11, 33, 35, 36, 37, 62, 63] and Van De Wouw, Leine [77, 78, 79, 118]. The solution always exists uniquely due to the fact that the operator involved is maximal monotone. The papers [5, 6, 8, 10, 35] give a Lyapunov theorem for stability and attractivity for first-order and linear second-order differential inclusions. Non-linear second-order differential inclusions are discussed in [9] via studying Lagrange dynamical systems with a set-valued controller. In this article, the well-posedness, stability analysis including finite-time stability are given by using a smooth Lyapunov function chosen from the particular properties of the systems.

In [33], Brogliato uses measure differential inclusions to describe mechanical system with frictionless impact, generalizing the Lagrange - Dirichlet stability theorem. More general works can be found in [46, 47] by Chareyron and Wieber. Furthermore, Lasalle's invariance principle is extended in [35] by Brogliato and Goeleven to evolution variational inequalities and to systems with unilateral constraints (in particular, mechanical systems with frictionless unilateral contacts).

In [61], Goeleven and Brogliato give instability results for finite dimensional variational inequalities and instability results for a class of parabolic variational inequalities in Hilbert space is given by Quittner [101]. It is know that in many cases, finding a smooth Lyapunov function is impossible. Hence, it is natural to develop the stability theory for non-smooth Lyapunov functions, see for examples the work of S. Adly, A. Hantoute et M. Théra [12, 13] as well as the following references [23, 24].

1.5 Objective and Outline of the Thesis

The main goal of this thesis is to provide a formulation to study the stability analysis of non-smooth dynamical systems, particularly in electrical circuits and mechanics with dry friction. The efficient tools which we have used are non-smooth analysis, Lyapunov stability theorem and non-smooth mathematical frameworks: complementarity and differential inclusions. In details, we use complementarity formalism to modelize some simple switch systems and differential inclusions to modelize a dc-dc buck converter, Lagrange dynamical systems and Lur'e systems. For each model, we are interested in the well-posedness, stability properties of trajectories, even finite-time stability or putting a control force to obtain finite-time stability, and finding numerical ways to simulate the systems. It is noted that the method used in this monograph can be applied to analyze the non-smooth dynamical systems from other fields.

The monograph is divided into 6 chapters which have a close connection with one another. In chapter 1, we give a short introduction and motivation for studying stability analysis of non-smooth dynamical systems with a concise suggested references. In chapter 2 we give the notations and necessary background to the reader. It concentrates on non-smooth analysis, Filippov theory and Lyapunov stability theory. Chapter 3 is dedicated to analyzing the electrical circuits. Firstly, the ampere-volt characteristics of some electrical devices, including some non-smooth devices like diodes, diacs, silicon controller rectifiers... are presented. Then, some simple RLC circuits with diodes are analyzed by the tool of complementarity formulation. At the end part of the chapter, a dc-dc buck converter are modeled by differential inclusion and their well-posedness, stability results are given. Chapter 4 is about Lagrange dynamical systems subject to a perturbation force and the control force of the inclusion form. In this chapter, the well-posedness and stability properties of the trajectories are analyzed carefully. The control force of the inclusion form put the trajectories converge to the equilibria in a finite time which can be estimated. In chapter 5, the stability analysis of general framework Lur'e systems are studied. The theoretical results are supported by some examples in electrical circuits with numerical simulations. Chapter 6 gives some conclusions about the work and several perspectives for further researches.

Chapter 2

Mathematical backgrounds

The main purpose of this chapter is to provide the reader with the basic notations and results used in convex analysis, non-smooth analysis, Filippov theory and Lyapunov stability theory. They are chosen carefully and written in an easily understandable manner so that people from other domains can also follow.

2.1 Convex Analysis

2.1.1 Convex Sets

Definition 2.1.1 *(Convex Set) A subset $C \subset \mathbb{R}^n$ is a* convex *set if for all $x, y \in C, [x, y] \subset C$, i.e. for any $x,\ y \in C$ and $\lambda \in [0, 1]$, we have:*

$$\lambda x + (1 - \lambda)y \in C.$$

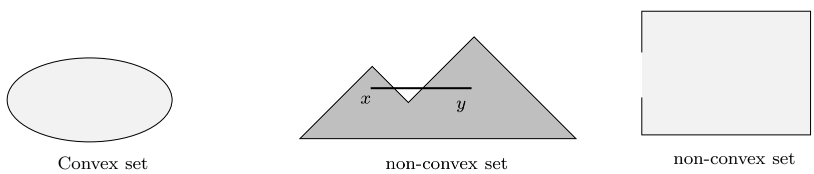


Figure 2.1: Convex and non-convex sets

Example 2.1.1

- *In $\mathbb{R}$, convex sets are exactly the intervals;*
- *In $\mathbb{R}^n$ an affine manifold is a convex set;*
- *The set Δ_n defined by:*

$$\Delta_n = \left\{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \ : \ \alpha_i \geq 0 \text{ for } i = 1, \ldots, n \text{ and } \sum_{i=1}^{n} \alpha_i = 1 \right\}$$

  is called the unit simplex in $\mathbb{R}^n$ and it is a convex set.

Proposition 2.1.1 *Let I be an arbitrary index subset and suppose that for every $i \in I$, C_i is a convex subset. Then $\bigcap_{i \in I} C_i$ is a convex subset.*

Definition 2.1.2 *(Convex Combination) A convex combination of the points* $(x_i)_{1\le i\le k} \subset \mathbb{R}^n$ *is a point of the form:*

$$x = \sum_{i=1}^{k} \lambda_i x_i, \quad \text{with } \lambda_i \ge 0,\ \forall i = 1, 2, \ldots, k \text{ and } \sum_{i=1}^{k} \lambda_i = 1,$$

i.e. x is is an affine combination of $x_1, \ \ldots, x_k$*, with all the scalars* $\lambda_i \ge 0$ *for* $i = 1, 2, \ldots, k$*. The convex hull of a set S, denoted by co(S), is the smallest convex set containing S in the following sense: if* $C \subset \mathbb{R}^n$ *is a convex set such that:* $S \subset C$*, then* $co(S) \subset C$*. It is also the intersection of all convex sets containing S.*

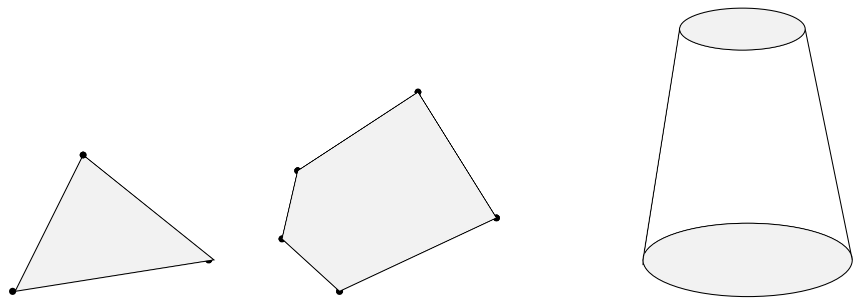


Figure 2.2: Convex hull of sets in $\mathbb{R}^2$

Remark 2.1.1 *C is a convex set if and only if* $\mathrm{co}(C) = C$*.*

Definition 2.1.3 *(Closed Convex Hull) The closed convex hull of a subset* $S \subset \mathbb{R}^n$*, denoted by* $\overline{\mathrm{co}}(S)$*, is the intersection of all closed convex sets containing S.*

Proposition 2.1.2 $\overline{\mathrm{co}}(S) = \overline{\mathrm{co}(S)}$.

Theorem 2.1.1 *(Caratheodory, 1907):*
Let $S \subset \mathbb{R}^n$ *be given. Then each element in* $\mathrm{co}(S)$ *can be written as a convex combination of at most* $(n+1)$ *elements in S, i.e.*

$$x \in \mathrm{co}(S) \iff \exists \lambda_i \ge 0,\ \exists x_i \in S,\ i = 1, 2, \ldots, n+1 \text{ such that } \sum_{i=1}^{n+1} \lambda_i = 1 \text{ and } x = \sum_{i=1}^{n+1} \lambda_i x_i.$$

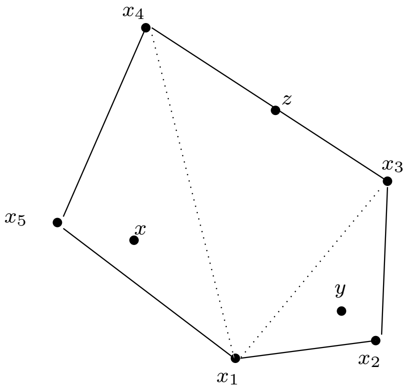


Figure 2.3: Illustration of Theorem 2.1.1 with $S = \{x_1, x_2, x_3, x_3, x_4, x_5\}$. The points x, y and z can be represented as a convex combination of the points $\{x_1, x_4, x_5\}$, $\{x_1, x_2, x_3\}$ and $\{x_4, x_3\}$ respectively.

2.1.2 Convex Cones

Definition 2.1.4 *(Cone) A subet C of $\mathbb{R}^n$ is called a cone if $\mathbb{R}_+^* C \subset C$, i.e.*

$$\forall x \in C,\ \forall \lambda > 0,\ \lambda x \in C.$$

Equivalently, C contains the open half-line $\mathbb{R}_+^* x$ for every $x \in C$. A convex cone C is a cone which is convex. A point of the form:

$$\sum_{i=1}^{k} \lambda_i x_i \text{ with } \lambda_i \in \mathbb{R}_+ \text{ for } i = 1, \ldots, k$$

is called a conic combination of the points $x_1, \ldots, x_k$.
The conic hull of $S \subset \mathbb{R}^n$, denoted by $\operatorname{cone}(S)$, is the smallest convex cone that contains S.

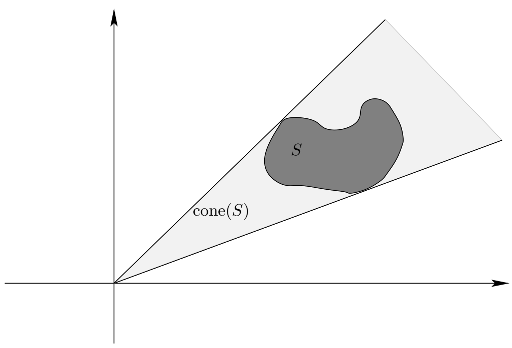


Figure 2.4: Conic hull of a subset S.

We have the following theorem:

Theorem 2.1.2 *Let E be a convex set. The set $\{\lambda x | x \in E, \lambda > 0\}$ is the conic hull of E.*

Theorem 2.1.3 *Let S be an non-empty set of $\mathbb{R}^n$. Then we have:*

$$\text{cone}(S) = \text{cone}(\text{co}(S)),$$

and the conic hull of S is the set of all conic combinations of points in S.
Each element in $\text{cone}(S)$ can be written as a conic combination of at most $(n+1)$ elements in S.

Definition 2.1.5 *(Dual Cone) Let C be a non-empty subset in $\mathbb{R}^n$. The* dual cone *C^* of a C in $\mathbb{R}^n$ is defined by:*

$$C^* = \{p \in \mathbb{R}^n : \langle x, p \rangle \geq 0, \forall x \in C\}.$$

Remark 2.1.2 *It is easy to show that the dual cone of a subset C coincides with the dual cone of the cone generated by C i.e.*

$$C^* = \big(\text{cone}(C)\big)^*.$$

If C is subspace of $\mathbb{R}^n$, then C^* coincides with the orthogonal subspace of C i.e.

$$C^* = C^{\perp} = \{p \in \mathbb{R}^n : \langle x, p \rangle = 0, \forall x \in C\}.$$

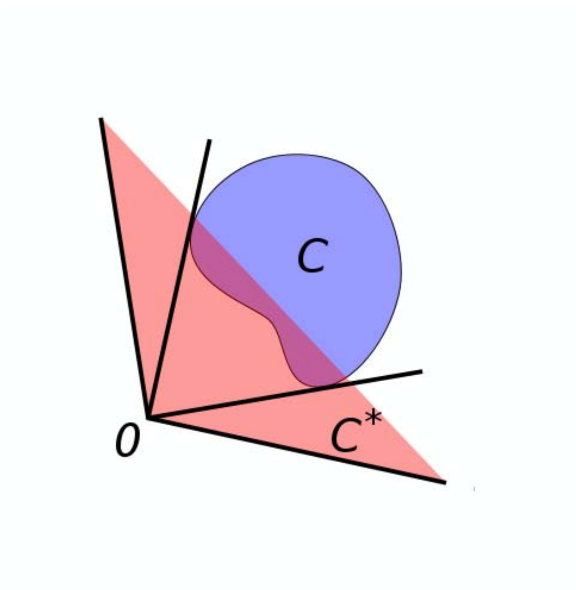


Figure 2.5: A set C and its dual cone C^*.

Definition 2.1.6 *(Polar Cone) The polar cone C° of a non-empty subset C in $\mathbb{R}^n$ is defined by:*

$$C^\circ = \{p \in \mathbb{R}^n \;:\; \langle x, p\rangle \leq 0, \forall x \in C\} = -C^*.$$

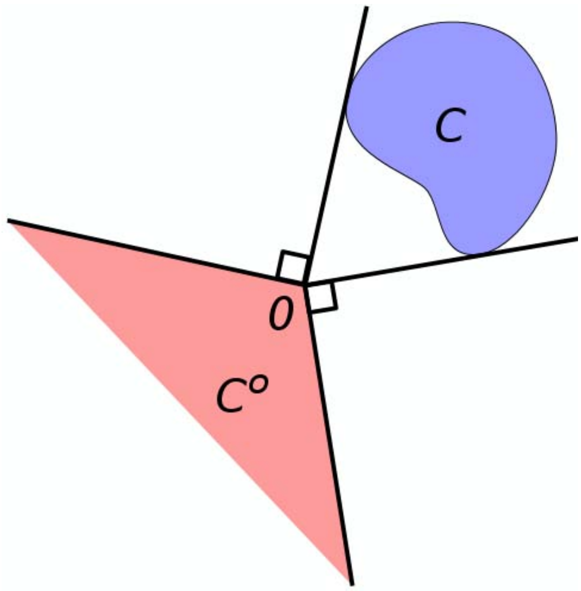


Figure 2.6: A set C and its polar cone C^o.

Definition 2.1.7 *(Normal Cone) The normal cone to a non-empty subset C in $\mathbb{R}^n$ at a point $x \in C$ is defined by:*

$$N_C(x) = \{p \in R^n : \langle p, y - x\rangle \leq 0, \forall y \in C\}.$$

If x belongs to the interior of C, then $N_C(x) = \{0\}$.

Definition 2.1.8 *(Tangent Cone) The tangent cone to a non-empty subset C in $\mathbb{R}^n$ at a point $x \in C$ is defined by:*

$$T_C(x) = \{y \mid \exists t_k \downarrow 0, y_k \to y, x + t_k y_k \in C\}.$$

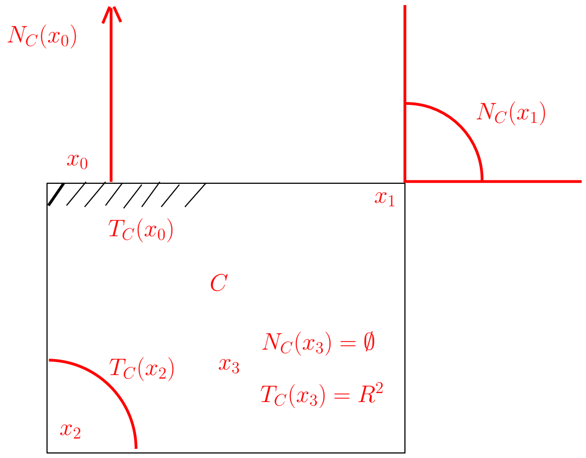


Figure 2.7: Normal cone and tangent cone of a convex set.

Remark 2.1.3 *If C is convex, the tangent cone to C at x is also the polar cone of the normal cone to C at x.*

2.1.3 Functions and Continuiaty

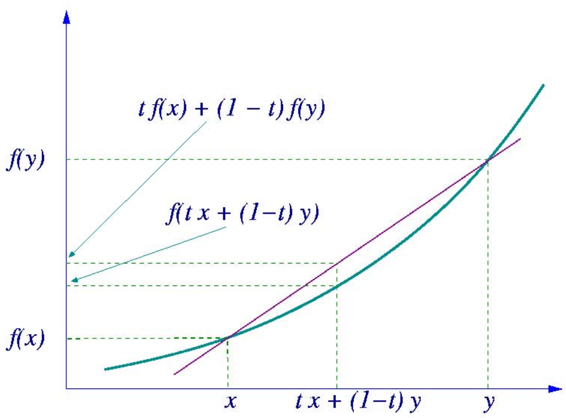


Figure 2.8: Convex function on an interval.

Given a function $f : S \to (-\infty, +\infty]$ on a sets $S \subset R^n$, the set:

$$\text{dom}(f) = \{x \in S \mid f(x) < +\infty\}$$
$$\text{epi}(f) = \{(x, \alpha) \in S \times R \mid f(x) \leq \alpha\}$$

are called the *effective domain* and *epigraph* of $f(x)$, respectively. If $domf \neq \emptyset$ then we say that the function $f(x)$ is *proper*.

Definition 2.1.9 *(Convex Function) A function $f : S \to (-\infty, +\infty]$ is called convex if its epigraph is a convex set in $\mathbb{R}^n \times \mathbb{R}$.*

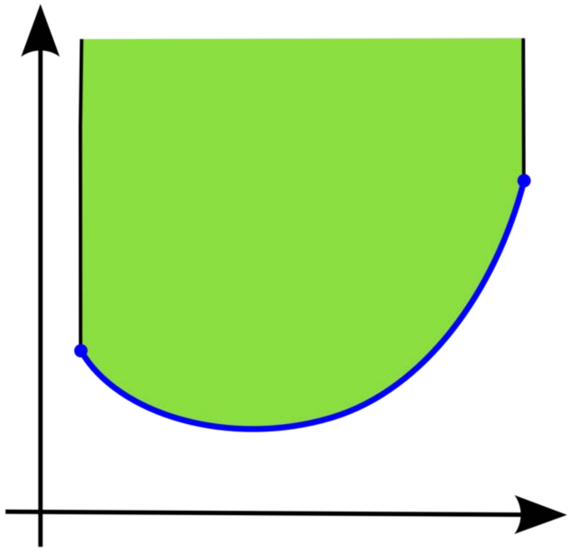


Figure 2.9: A function (in blue) is convex if and only if the region above its graph (in green) is a convex set.

This is equivalent to say that S is a convex set in $\mathbb{R}^n$ and for $x, y \in S$ and $\lambda \in [0, 1]$, we have

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y).$$

A function $f(\cdot)$ is said to be *concave* on S if $-f(\cdot)$ is convex; *affine* on S if $f(\cdot)$ is finite and both convex and concave. It's easy to see that an affine function on $\mathbb{R}^n$ has the form $f(x) = \langle a, x \rangle + \alpha$ with $a \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$.

For a given nonempty convex set $C \in \mathbb{R}^n$, we can define the convex functions:

- the *indicator function* of C: $\delta_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{if } x \notin C. \end{cases}$
- the *support function*: $\sigma_C(x) = \sup_{y \in C} \langle y, x \rangle$.
- the *distance function*: $d(x, C) = \inf_{y \in C} \|x - y\|$.

For a function $f : \mathbb{R}^n \to (-\infty, +\infty]$, we can define semi-continuity:

Definition 2.1.10 *(Upper Semi-continuity) A function $f(\cdot)$ is called upper semi-continuous if*

$$\limsup_{y \to x} f(y) \leq f(x), \forall x \in \mathbb{R}^n.$$

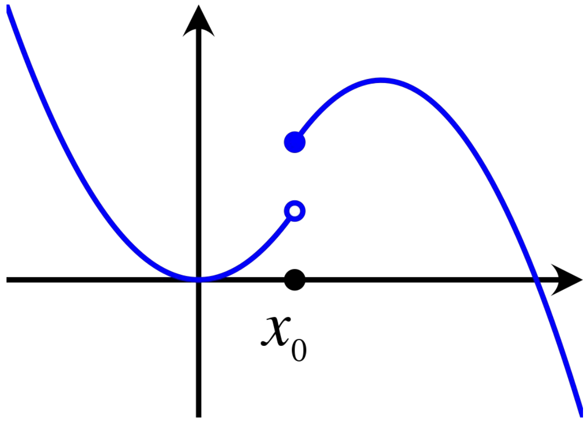


Figure 2.10: An upper semi-continuous function f. The solid blue dot indicates $f(x^0)$.

We can see that f is upper semi-continuous iff for every $\epsilon > 0$ there exists a neighborhood V s.t $f(y) < f(x) + \epsilon$ for all $y \in V$. Similarly, lower semi-continuous function is introduced.

Definition 2.1.11 *(Lower Semi-continuity) A function $f(\cdot)$ is called lower semi-continuous if*

$$\lim_{y \to x} \inf f(y) \geq f(x), \forall x \in \mathbb{R}^n.$$

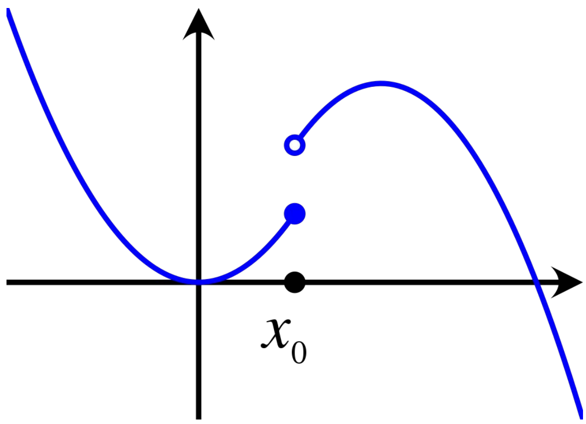


Figure 2.11: An lower semi-continuous function f.

If $f(\cdot)$ is lower semi-continuous then $-f(\cdot)$ is upper semi-continuous. So, f is lower semi-continuous iff for every $\epsilon > 0$ there exists a neighborhood V s.t $f(y) > f(x) - \epsilon$ for all $y \in V$. A function which is both upper semi-continuous and lower semi-continuous is called continuous. A stronger concept than continuity is absolute continuity:

Definition 2.1.12 *(Absolute Continuity) A function $f : I \to \mathbb{R}^n$ is called absolute contin-*

uous on $I \subset \mathbb{R}$ if for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that:

$$\sum_{i=1}^{n} \|f(b_i) - f(a_i)\| < \epsilon,$$

for any n and any disjoint collection of intervals $[a_i, b_i] \in I$ satisfying:

$$\sum_{i=1}^{n} (b_i - a_i) < \delta.$$

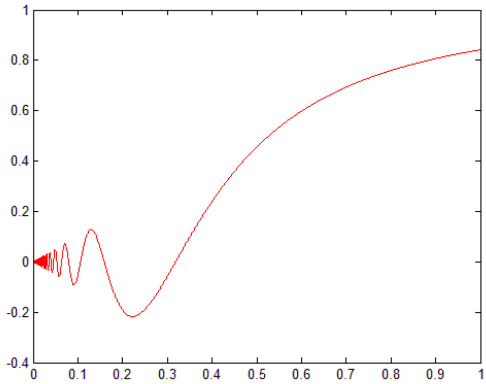


Figure 2.12: The function f with $f(x) = xsin(\frac{1}{x})$ *for* $x \in (0, +\infty)$ *and* $f(0) = 0$ is continuous but not bounded and not absolutely continuous.

If f is Lipschitz-continuous then f is obviously absolutely continuous. Let I be a real interval and X be a Euclidean space with the norm $\|\cdot\|$.

Definition 2.1.13 *(Variation) Let $f : I \to X$ and let $[a, b]$ be a subinterval of I. The variation of f on $[a, b]$ is the nonnegative extended real number:*

$$var(f, [a, b]) = \sup \sum_{i=1}^{n} \|f(x_i) - f(x_{i-1})\|$$

where the supremum is taken over all strictly increasing finite sequences $x_1 < x_2 < \ldots < x_n$ of points on $[a, b]$.

Definition 2.1.14 *(Locally Bounded Variation) The function $f : I \to X$ is said to be of locally bounded variation, $f \in lbv(I, X)$, if and only if:*

$$var(f, [a, b]) < \infty$$

for every compact subinterval $[a, b]$ of I.

If $f \in lbv(I, X)$ then there exist a right limit $f^+(x)$ and a left limit $f^-(x)$ at each $x \in I$:

$$f^+(x) = \lim_{y \downarrow x} f(y), \qquad f^-(x) = \lim_{y \uparrow x} f(y).$$

If f is continuous at x, then we obtain $f(x) = f^+(x) = f^-(x)$.

Definition 2.1.15 *(Conjugate Function) Let f be a proper convex function. The function f^* is called the conjugate function of f and is defined as:*

$$f^*(x^*) = \sup_x \{x^T x^* - f(x)\}.$$

Note that the conjugate of indicator function is the support function. The conjugate f^* of a convex function is again convex. If we take the conjugate of f^*, then we get the original function.

Theorem 2.1.4 *(Fenchel-Moreau) If f is a lower semi-continuous convex function, then it holds that $f^{**} = f$.*

From the definition, we have the Fenchel's inequality

$$x^T x^* \leq f(x) + f^*(x^*).$$

The equality holds $\Leftrightarrow x^* \in \partial f(x) \Leftrightarrow x \in \partial f^*(x^*)$ where $\partial(\cdot)$ denotes the subdifferential in convex analysis.

2.2 Non-smooth Analysis

2.2.1 Generalized Derivatives

The classical derivative of smooth continuous functions will be extended to the generalized derivative (and differential) of Clarke for non-smooth lower semi-continuous functions.

Definition 2.2.1 *(Generalized Differential) For $f : \mathbb{R}^n \to \mathbb{R}$, locally Lipschitz-continuous, the generalized differential of Clarke is defined as:*

$$\partial f(x) = \overline{co}\{\lim_{i \to \infty} \nabla f(x_i) : x_i \to x, \nabla f(x_i) \ \textit{exists}\} \subset R^n, \tag{2.1}$$

with the gradient:

$$\nabla f(x) = (\frac{\partial f(x)}{\partial x})^T \subset R^n.$$

The generalized differential can be equivalently defined as:

$$\partial f(x) = \{w \in \mathbb{R}^n : f^0(x; v) \geq \langle w, v \rangle \ \ \forall v \in \mathbb{R}^n\},$$

where $f^0(x; v)$ is the generalized directional derivative of f at x in the direction v defined by:

$$f^0(x; v) := \limsup_{y \to x, t \to 0^+} \frac{f(y + tv) - f(v)}{t}.$$

If f is proper lower semi-continuous convex functions then its generalized differential agrees with the subdifferential.

Definition 2.2.2 *(Subdifferential) For a lower semi-continuous convex function* $f : \mathbb{R}^n \to (-\infty, +\infty]$*, the subdifferential of* f *at* $x \in \mathbb{R}^n$ *with* $f(x) < +\infty$ *is defined as:*

$$\partial f(x) = \{p \mid f(x^*) \geq f(x) + p^T(x^* - x),\ \forall x^* \in \mathbb{R}^n\} \subset \mathbb{R}^n.$$

If $p \in \partial f(x)$ *then* p *is called a subgradient of* f *at the point* x*.*

Example 2.2.1 *Consider* $f : \mathbb{R} \to \mathbb{R}, x \to |x|$*. Then:*

$$\partial f(x) = \begin{cases} 1 & x > 0, \\ [-1, 1] & x = 0, \\ -1 & x < 0. \end{cases}$$

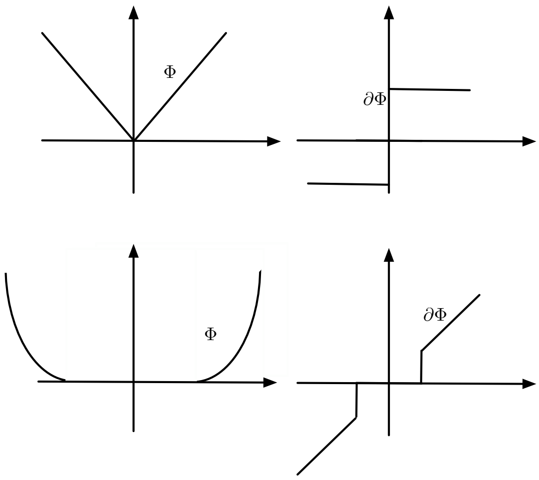


Figure 2.13: Examples of subdifferential.

Denote by $\Gamma_0(H)$ the set of all proper lower semi-continuous convex functions on H.

Theorem 2.2.1 *Let* $f \in \Gamma_0(\mathbb{R}^n)$*. Then* $\partial f(x)$ *is a closed convex, possibly empty. But if* $x \in \mathrm{Int}(Dom f)$*, then* $\partial f(x) \neq \emptyset$*.*

In particular, if $f : \mathbb{R}^n \to \mathbb{R}$ *is convex, then for all* $x \in \mathbb{R}^n$*,* $\partial f(x)$ *is a nonempty, convex and compact set of* $\mathbb{R}^n$*.*

Proposition 2.2.1 *([29],[99]) Let* $f(\cdot)$ *be a convex function from* $\mathbb{R}^n$ *to* $\mathbb{R}$*. Then:*
1) $f(\cdot)$ *is locally Lipschitz.*
2) $\partial f(\cdot)$ *is maximal monotone and bounded on bounded sets.*
3) $\partial f(\cdot)$ *is upper semicontinuous with nonempty, convex and compact values.*

Theorem 2.2.2 *Let* $f, g \in \Gamma_0(\mathbb{R}^n)$ *and suppose* f *is continuous at a point* $x_0 \in Dom(g)$*. Then for all* $x \in Dom(f + g)$*, we have:*

$$\partial(f + g)(x) = \partial f(x) + \partial g(x).$$

Proposition 2.2.2 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $A \in \mathbb{R}^{n\times m}$ be a matrix. We define the $F : \mathbb{R}^m \to \mathbb{R}, x \to f(Ax)$. Then:*

$$\partial F(x) = A^T \partial f(Ax) = \{A^T p \ : \ p \in \partial f(Ax)\}.$$

Proposition 2.2.3 *(Gronwall's inequality) Let $T > 0$ be given and let $b \in L^1([t_0, t_0+T], \mathbb{R})$. Let the absolutely continuous function $u :\ [t_0, t_0+T] \to \mathbb{R}$ satisfy:*

$$\dot{u}(t) \le b(t)u(t),\ a.e\ t \in [t_0, t_0+T].$$

Then:

$$u(t) \le u(t_0)\exp\left(\int_{t_0}^{t} b(s)ds\right) \text{ for all } t \in [t_0, t_0+T].$$

Definition 2.2.3 *Given $1 \le k, p \le \infty$. The Sobolev space $\mathcal{W}^{k,p}([0,T];\mathbb{R}^n)$ is defined by:*

$$\mathcal{W}^{k,p}([0,T];\mathbb{R}^n) := \{u \in L^p([0,T];\mathbb{R}^n) : u', \ldots, u^{(k)} \textit{ exist and belong to } L^p([0,T];\mathbb{R}^n)\},$$

where the derivatives $u', \ldots, u^{(k)}$ are understood in the weak sense.

2.2.2 Monotone Operators

Let H is a Hilbert space and its dual $H^* \equiv H$.

Definition 2.2.4 *(Set-valued Functions) A set-valued function $\mathcal{F} : H \rightrightarrows H$ is a map that associates with any $x \in H$ a set $\mathcal{F}(x) \subset H$.*

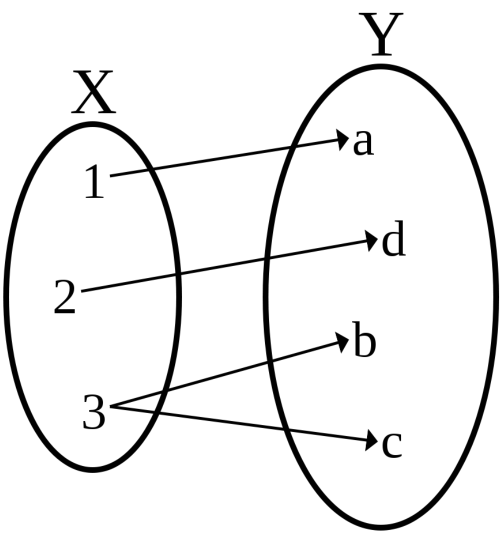


Figure 2.14: Set-valued function.

The graph of the set-valued function $\mathcal{F}$ is defined by:

$$Graph(\mathcal{F}) = \{(x, y) \in H \times H \mid y \in \mathcal{F}(x)\}.$$

Set-valued function is sometimes called multi-valued function or multifunction.

Definition 2.2.5 *(Convex image) A set-valued function $\mathcal{F}$ has a convex image on $X \subset H$ if the image of x under $\mathcal{F}$ is a convex set for all fixed values $x \in X$.*

Definition 2.2.6 *(Upper Semi-continuity of Set-valued Functions) A set-valued function $\mathcal{F}$ is upper semi-continuous in x if:*

$$\lim_{y \to x} (\sup_{a \in \mathcal{F}(y)} \inf_{b \in \mathcal{F}(x)} \|a - b\|) \to 0.$$

This condition is equivalent to the condition that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x - y\| < \delta \Rightarrow \mathcal{F}(y) \subset \mathcal{F}(x) + B_\varepsilon$. It is also equivalent to the following definition:

Definition 2.2.7 *Let $\mathcal{F} : H \rightrightarrows H$ be a set-valued function. One says that $\mathcal{F}(\cdot)$ is upper semi-continuous at $x_0 \in \mathbb{R}^n$ if for any open neighborhood $\mathcal{N}$ containing $\mathcal{F}(x_0)$ there exists an open neighborhood $\mathcal{M}$ of x_0 such that $\mathcal{F}(\mathcal{M}) \subset \mathcal{N}$.*

Proposition 2.2.4 *([19] p.41) Let $F_1 : H \rightrightarrows H$ and $F_2 : \mathbb{R}^m \rightrightarrows \mathbb{R}^q$ be two set-valued functions. Define $F_2 \circ F_1 : \mathbb{R}^n \rightrightarrows \mathbb{R}^q$ by:*

$$(F_2 \circ F_1)(x) := \bigcup_{y \in F_1(x)} F_2(y).$$

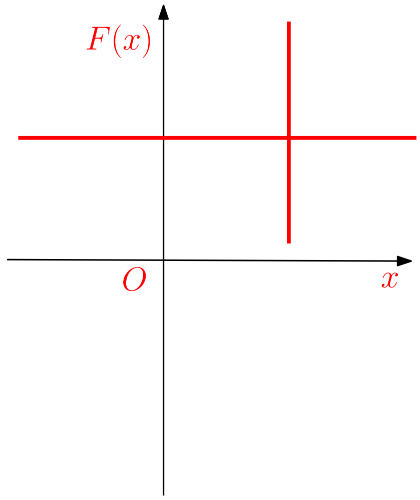
If $F_1(\cdot)$ and $F_2(\cdot)$ are upper semicontinuous then $(F_2 \circ F_1)(\cdot)$ is upper semicontinuous. In particular, $(F_1 + F_2)(\cdot)$ is upper semicontinuous.

Proposition 2.2.5 *([55] p.66) Let a set D be closed, and a set-valued function $F(\cdot)$ with closed values be bounded in a neighborhood of each point $p \in D$. Then the function $F(\cdot)$ is upper semicontinuous on D if and only if its graph is closed.*

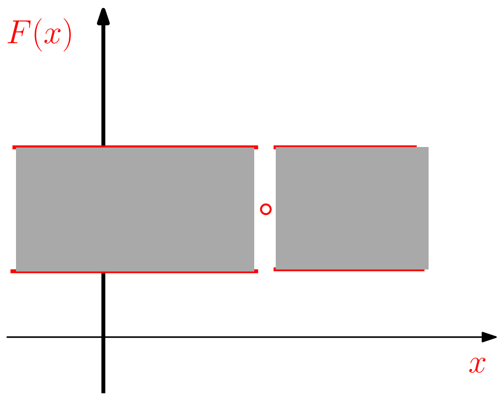
Definition 2.2.8 *(Lower Semi-continuity of Set-valued Functions) A set-valued function $\mathcal{F}$ is lower semi-continuous in x if:*

$$\lim_{y \to x} (\inf_{a \in \mathcal{F}(y)} \sup_{b \in \mathcal{F}(x)} \|a - b\|) \to 0.$$

This condition is equivalent to the condition that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x - y\| < \delta \Rightarrow \mathring{A}\mathcal{F}(x) \subset \mathcal{F}(y) + B_\varepsilon$. Upper and lower semi-continuity of set-valued functions are sometimes called outer and inner semi-continuity.



Upper semi-continuous but not lower semi-continuous



Lower semi-continuous, not upper semi-continuous

Figure 2.15: Upper semi-continuity and lower semi-continuity of a set-valued function.

Example 2.2.2 *The set-valued mapping* $\mathrm{Sign} : \mathbb{R} \rightrightarrows \mathbb{R}; z \to \mathrm{Sign}(z)$ *is upper semi-continuous.*
Indeed, let $z > 0$ *then* $\mathrm{Sign}(z) = \{1\}$ *and for any open neighborhood* $\mathcal{M}$ *of* $\{1\}$*, there exists* $\delta > 0$ *sufficiently small so that* $]z-\delta, z+\delta[\subset]0, +\infty[$ *and thus* $F(]z-\delta, z+\delta[) = \{1\} \subset \mathcal{M}$*.*
If $z < 0$ *then* $\mathrm{Sign}(z) = \{-1\}$ *and for any open neighborhood* $\mathcal{M}$ *of* $\{-1\}$*, there exists* $\delta > 0$ *sufficiently small so that* $]z-\delta, z+\delta[\subset]-\infty, 0[$ *and thus* $F(]z-\delta, z+\delta[) = \{-1\} \subset \mathcal{M}$*.*
Finally, if $z = 0$ *then* $\mathrm{Sign}(z) = [-1, +1]$ *and for any open neighborhood* $\mathcal{M}$ *of* $[-1, +1]$ *and for any* $\delta > 0$ *we have* $F(]-\delta, +\delta[) = [-1, +1] \subset \mathcal{M}$*.*

The concept of measurability has also been generalized to set-valued function as follows (see [43]). In this paper, we consider the Lebesgue measure on $\mathbb{R}_+$.

Definition 2.2.9 *Let* $U : [0, +\infty[\rightrightarrows \mathbb{R}^n$ *be a set-valued map with closed nonempty images.*

One says that U is measurable provided that for every open set $\mathcal{O} \subset \mathbb{R}^n$ the set $U^-(\mathcal{O}) = \{t \in [0, +\infty[: U(t) \cap \mathcal{O} \neq \emptyset\}$ is measurable.

Theorem 2.2.3 *[43] Let $U : [0, +\infty[\rightrightarrows \mathbb{R}^n$ be a set-valued map with closed nonempty images. Then U is measurable if and only if $\forall z \in \mathbb{R}^n$, the single-valued mapping $t \mapsto d(z, U(t))$ is measurable.*

Definition 2.2.10 *(Monotone Set-valued Function) A set-valued function $\mathcal{F}$ is called monotone if its graph is monotone in the sense that:*

$$\forall (x,y) \in Graph(\mathcal{F}),\ \ \forall (x^*,y^*) \in Graph(\mathcal{F}),\ \ \langle y - y^*, x - x^* \rangle \geq 0.$$

$\mathcal{F}(x)$ is not required to be nonempty. The set $D(\mathcal{F}) = \{x \in H : \mathcal{F}(x) \neq \emptyset\}$ is called the domain (or effective domain) of $\mathcal{F}$.

Example 2.2.3 *(i) If $\mathcal{F}$ is a linear map, then it is monotone iff it is a positive operator: $\langle \mathcal{F}(x), x \rangle \geq 0$ for all x.*

(ii) Let A be a nonempty subset of $\mathbb{R}$. A function $\varphi : A \to \mathbb{R}$ is monotone iff:

$$[\varphi(t_1) - \varphi(t_2)](t_1 - t_2) \geq 0\ \ \forall\, t_1, t_2 \in A.$$

(iii) Let $\varphi(x) = -1$ if $x < 0$, $\varphi(x) = 1$ if $x > 0$ and $\varphi(0)$ be any subset of $[-1, 1]$, then φ is a set-valued monotone function.

(iv) The following example is an important single-valued but nonlinear one : Let φ be a continuous real-valued function on H and φ is Gateaux differentiable, i.e, for given $x \in H$, for all $y \in H$, the limit:

$$d\varphi(x)(y) = \lim_{t \to 0} \frac{\varphi(x + ty) - \varphi(x)}{t}$$

exists and is a bounded linear functional of y. If φ is convex then the mapping $x \to d\varphi(x)$ is monotone. In fact, for $0 < t < 1$, we have:

$$\frac{\varphi(x + t(y - x)) - \varphi(x)}{t} \leq \frac{(1 - t)\varphi(x) + t\varphi(y) - \varphi(x)}{t} = \varphi(y) - \varphi(x).$$

It implies that $d\varphi(x)(y - x) \leq \varphi(y) - \varphi(x)\ \ \forall x, y \in E$. Let $x^* = d\varphi(x), y^* = d\varphi(y)$, then:

$$\langle x^*, y - x \rangle \leq \varphi(y) - \varphi(x)\ \ \text{and}\ \ -\langle y^*, y - x \rangle = \langle y^*, x - y \rangle \leq \varphi(x) - \varphi(y).$$

Adding these two inequalities, we obtain the desired result.

(v) Here is an example which arises in fixed-point theory. Let A be a bounded closed convex nonempty subset of H and let T be a nonexpansive map of A into itself (i.e, $\|T(x) - T(y)\| \leq \|x - y\|\ \forall\, x, y \in A$). Then $\mathcal{F} = I - T$, where I is the identity map in H, is monotone with $D(T) = A$. Indeed, for all $x, y \in A$:

$$\langle F(x) - F(y), x - y \rangle = \langle x - y - (T(x) - T(y)), x - y \rangle$$

$$= ||x-y||^2 - \langle T(x)-T(y), x-y\rangle \geq ||x-y||^2 - ||T(x)-T(y)|| \cdot ||x-y|| \geq 0.$$

(vi) *Let A be a nonempty closed convex set of H and let P be the metric projection of H onto A (i.e, $P(x)$ is the unique element of A such that $\|x-P(x)\| = \inf\{\|x-y\| : y \in A\}$). Then, we have the fundamental variational inequality: for all $x \in H$,*

$$\langle x-P(x), z-P(x)\rangle \leq 0, \ \ \forall z \in A.$$

Let $z = P(y)$, we obtain:

$$\langle x-P(x), P(y)-P(x)\rangle \leq 0.$$

Similarly,

$$\langle y-P(y), P(x)-P(y)\rangle \leq 0.$$

Adding these two inequalities, we have

$$\langle x-y, P(x)-P(y)\rangle \geq \|P(x)-P(y)\|^2 \ \ \forall x, y \in H.$$

It means that P is monotone in a strong sense.

Definition 2.2.11 *(Maximal Monotone Set-valued Function) A monotone set-valued function $\mathcal{F}$ is called maximal monotone if there exists no other monotone set-valued function whose graph strictly contains the graph of $\mathcal{F}$.*

An application of Zorn's lemma implies that every monotone operator $\mathcal{F}$ can be extended to a maximal monotone operator $\bar{\mathcal{F}}$, in the sense that $Graph(\mathcal{F}) \subset Graph(\bar{\mathcal{F}})$. The following proposition may be considered as the mathematical expression for the maximality of a monotone operator. It is also a useful criterion to check whether a point u belongs to the set $\mathcal{F}(x)$.

Proposition 2.2.6 *The set-valued operator $\mathcal{F}$ is maximal monotone if and only if the two following statements are equivalent:*
(a) For every $(y, v) \in Graph(F)$, $\langle u-v, x-y\rangle \geq 0$
(b) $u \in \mathcal{F}(x)$.

Next, we give some nice properties of the image and graph of a maximal monotone operator.

Proposition 2.2.7 *Let $\mathcal{F}$ be a maximal monotone operator. Then:*
(a) Its images are closed and convex.
(b) Its graph is strongly-weakly closed, i.e., if $x_n \to x$ strongly and $u_n \in \mathcal{F}(x_n)$ converges weakly to u, then $u \in \mathcal{F}(x)$.

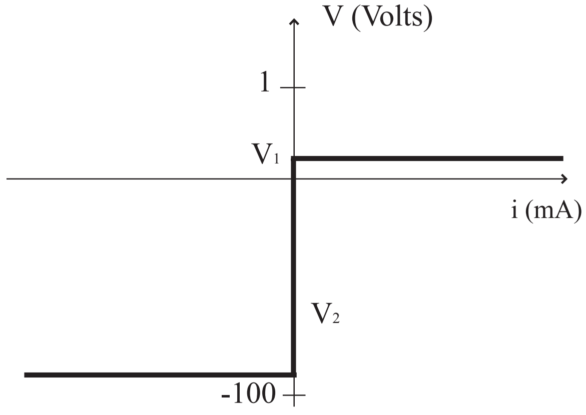


Figure 2.16: Maximal monotone set-valued function − diode model.

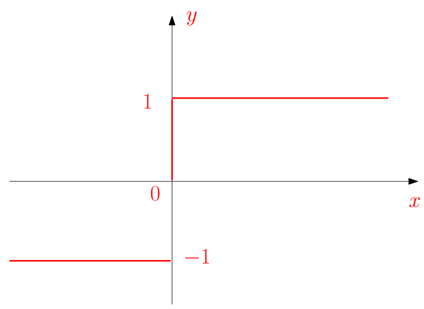


Figure 2.17: Monotone but not maximal monotone set-valued function .

Two following maximal monotone set-valued functions play a very important role in non-smooth analysis. The first one is the *unilateral primitive* , also the subdifferential of the indicator function:

$$\partial\Psi_{\mathbb{R}^+}(x) = N_{\mathbb{R}_+}(x) = \begin{cases} 0 & x > 0, \\ (-\infty, 0] & x = 0, \\ \emptyset & x < 0. \end{cases}$$

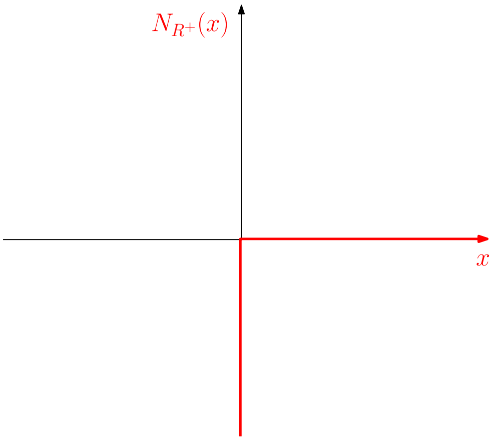


Figure 2.18: Unilateral primitive.

The other is set-valued Sign-function:

$$\text{Sign }(x) = \partial|x| = \begin{cases} 1 & x > 0, \\ [-1, 1] & x = 0, \\ -1 & x < 0. \end{cases}$$

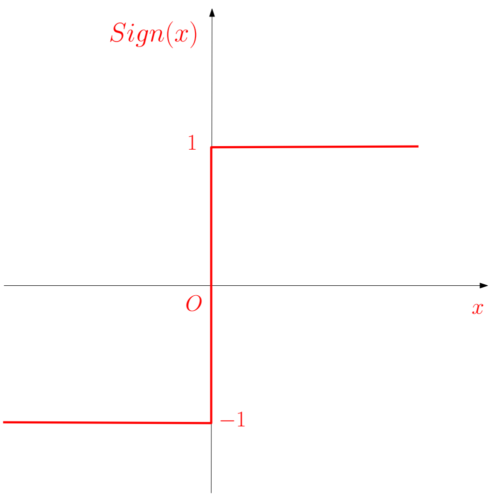


Figure 2.19: Sign function.

Definition 2.2.12 *(Inverse Operator) If $\mathcal{F} : H \rightrightarrows H$ is monotone, its inverse $\mathcal{F}^{-1}$ is the set valued mapping from $H \rightrightarrows H$ defined by $\mathcal{F}^{-1}(x^*) = \{x \in H : x^* \in \mathcal{F}(x)\}$. Then:*

$$Graph(\mathcal{F}^{-1}) = \{(x^*, x) \in H \times H : x^* \in \mathcal{F}(x)\}$$

and therefore, it is obvious that $\mathcal{F}^{-1}$ is maximal monotone iff $\mathcal{F}$ is maximal monotone.

Theorem 2.2.4 *(J. J. Moreau). If f is a proper lower semicontinuous convex function on a H, then its subdifferential ∂f is a maximal monotone operator.*

Theorem 2.2.5 *(Rockafellar).Suppose that $\mathcal{F}$ and $\mathcal{G}$ are maximal monotone operators on H and that $D(\mathcal{F}) \cap \text{int}D(\mathcal{G}) \neq \emptyset$. Then $\mathcal{F} + \mathcal{G}$ is maximal.*

Definition 2.2.13 *A set-valued map $\mathcal{F} : H \rightrightarrows H$ is said to be n-cyclically monotone iff:*

$$\sum_{1 \leq k \leq n} \langle x_k^*, x_k - x_{k-1} \rangle \geq 0$$

whenever $n \geq 2$ and $x_0, x_1, x_2, ..., x_n \in H, x_n = x_0$, and $x_k^* \in \mathcal{F}(x_k), k = 1, 2, 3, ..., n$. The mapping $\mathcal{F}$ is called cyclically monotone if it is n-cyclically monotone for every n. Clearly, a 2-cyclically monotone operator is monotone.

Example 2.2.4 *Let f be a proper lower semicontinuous convex function; then ∂f is cyclically monotone.*

Theorem 2.2.6 *(Rockafellar). If $\mathcal{F} : H \rightrightarrows H$ is maximal monotone and cyclically monotone, with $D(\mathcal{F}) \neq \emptyset$, then there exists a proper convex lower semicontinuous function f on H such that $T = \partial f$.*

Theorem 2.2.7 *(Minty). If $\mathcal{F} : H \rightrightarrows H$ is monotone. Then, $\mathcal{F}$ is maximal monotone iff $R(\mathcal{F} + I) = H$, i.e.,*

$$\forall f \in H, \exists x \in H \;\; s.t \;\; f \in x + \mathcal{F}(x).$$

Definition 2.2.14 *A set-valued map $\mathcal{F} : \; H \rightrightarrows H$ is called* hypomonotone *provided that there exists a real $k > 0$ such that for all $x_1, x_2 \in H$, we have:*

$$\langle \mathcal{F}(x_1) - \mathcal{F}(x_2), x_1 - x_2 \rangle \geq -k\|x_1 - x_2\|^2.$$

The map $\mathcal{F}(\cdot)$ is called locally hypomonotone *if for each $x_0 \in X$, $\mathcal{F}(\cdot)$ is hypomonotone in a neighborhood of x_0.*

2.3 Differential Inclusion, Filippov Theory and Lyapunov Theory

Roughly speaking, a differential inclusion (DI) is similar to an ordinary differential equation (ODE), but its right-hand side is a set-valued function instead of a single value function. Hence, it is a generalization of the concept of ODE of the form:

$$\dot{x}(t) \in \mathcal{F}(t, x(t)) \tag{2.2}$$

where $\mathcal{F} : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a set-valued map. The definition, existence, and uniqueness of solutions for differential inclusions is obviously more complicated than for ODEs and strongly depends on the boundary conditions, naturally on the regularity of the function $x(t)$ together with the set-valued map $\mathcal{F}$. One may have intuition that proving the existence of solutions for DIs is easier than ODEs since the right-hand side in DIs is "bigger" than the one in ODEs. Indeed, it is not true since we do not have the convergence of the derivatives in approximation, which is one of the main obstacles for DIs that does not exists for ODEs (Cellina [44]). In standard books on differential inclusions , the solution $x(t)$ is usually assumed to be absolutely continuous (AC) in time, i.e there are no discontinuities in the state $x(t)$. The cases of Lipschitzian and upper semi-continuous right-hand sides are the most common in the literature.

There is a great many of motivations that led mathematicians to study differential inclusions. For example, consider a standard system in control theory:

$$\dot{x}(t) = f(t, x(t), u(t)), \tag{2.3}$$

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and the control function u takes values in some prescribed subset U of $\mathbb{R}^m$. Taking $\mathcal{F}(t, x) := f(t, x, U)$, we obtain the differential inclusion (2.2). Under mild hypotheses on f, Filippov's Lemma implies that an AC function x satisfies (2.2) if and only if there is a measurable function u with values in U such that (2.3) holds. For another example, let us consider a simple second order system:

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + y = b.$$

Set $x_1 = y, x_2 = dy/dt$ and $x = (x_1\ x_2)^T$, we obtain the state equation:

$$\dot{x}(t) = f(t, a, b) \tag{2.4}$$

where f is a vetor-valued function. Suppose that the parameters a and b are uncertain and we only know the corresponding intervals where their values may belong. Note that we do not know about a possible probability distribution of these parameters and thus we cannot consider them as random variables. Therefore, the equation (2.4) takes the form of (2.2) where $\mathcal{F}$ is determined by the right-hand side of (2.4). We are interested in differential inclusions in order to study the existence, uniqueness and other properties of solutions of the dynamical systems with discontinuous right-hand side, for example.

Let us consider the following examples to make thing more clear:

Example 2.3.1 *Consider the differential equation with discontinuous right-hand side:*

$$\dot{x} = f(x) = 2 - 3\text{sign}(x). \tag{2.5}$$

For a given initial condition, we obtain a solution of (2.5)*:*

$$x(t) = \begin{cases} 5t + C_1, & x < 0, \\ -t + C_2, & x > 0, \end{cases}$$

where constants C_1 and C_2 are determined by the initial condition. Each solution reaches $x = 0$ in finite time. When it reaches $x = 0$, it cannot leave here since $\dot{x} > 0$ for $x < 0$ and $\dot{x} < 0$ for $x > 0$, i.e the solution stays at $x = 0$. But $x(t) = 0$ with $\dot{x} = 0$ is not a solution of (2.5) *since $0 \neq 2 - 3\text{sign}(0)$, i.e* (2.5) *have no solution in the classical sense. Naturally, one way to extend the notion of solution is to replace f by a set-valued $\mathcal{F}$ with*

$\mathcal{F}(x) = f(x)$ at all points where f is continuous. At the points where f is discontinuous, we choose appropriately for $\mathcal{F}(x)$. For instance, the differential equation (2.5) *can be replaced by the following DI:*

$$\dot{x} \in 2 - 3\text{Sign}(x). \tag{2.6}$$

Example 2.3.2 *Consider the model of Fig.* 2.20 *, the motion of system is governed by:*

$$m\ddot{x}(t) = mg\sin(\alpha) + F_{ext}(t) - \mu mg\cos(\alpha)\text{sign}(\dot{x}(t)),$$

where F_{ext} is the external force.

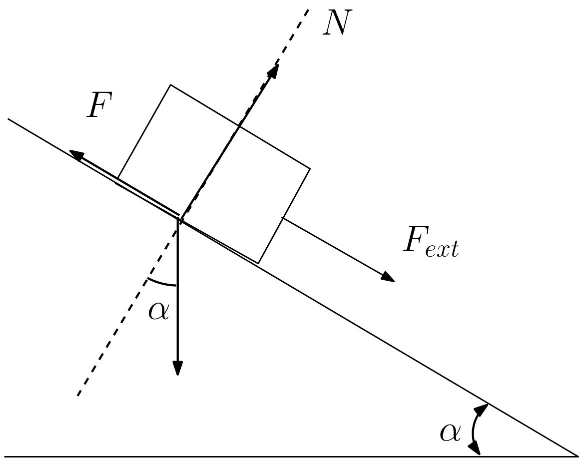


Figure 2.20: A mass on a inclined chute.

It is fact that there exists $\tau > 0$ such that $\dot{x}(t) = 0$, $\forall t > \tau$. If we set sign$(0) = 0$, then we have :

$$mg\sin(\alpha) + F_{ext}(t) = 0 \ \ \forall t > \tau.$$

But it is impossible. Therefore, we have to consider the differential conclusion:

$$m\ddot{x}(t) \in mg\sin(\alpha) + F_{ext}(t) - \mu mg\cos(\alpha)\text{Sign}(\dot{x}(t)).$$

The above examples show that differential equation with discontinuous right-hand side may have no solutions. However, we face discontinuous differential equation in a large number of applications. Many problems from mechanics and electrical engineering leads to differential equation with discontinuous right-hand side because many physical laws are expressed by discontinuous functions, for instance, a dry friction force or jump-like transition characteristic of some electronic devices. Many differential equations in control theory model objects with variable structure or with sliding motions and are discontinuous. Therefore, we have to define a new type of solution or more exactly to change slightly model to ensure the existence of solutions in simple cases for any given initial condition and coincide with solutions of the original model with continuous right-hand side. These requirements are fulfilled by *Filippov's convex method*, introduced in 1988. Filippov consider the set-valued map $\mathcal{F}(\cdot)$ with $\mathcal{F}(x)$ defined by:

$$\mathcal{F}(x(t)) = \bigcap_{\varepsilon>0} \bigcap_{\mu(N)=0} \overline{\text{co}} f((x(t) + \varepsilon\mathbb{B}_n)\backslash N). \tag{2.7}$$

where $\mathbb{B}_n$ denotes the unit ball of $\mathbb{R}^n$, the sets N are Lebesgue negligible and $\overline{\text{co}}$ is the closure of the convex hull. Since the graph of $\mathcal{F}(\cdot)$ is closed, if f is assumed to be linearly bounded, then the map $\mathcal{F}(\cdot)$ is upper semi-continuous. Furthermore, $\mathcal{F}(x) = \{f(x)\}$ whenever f is continuous.

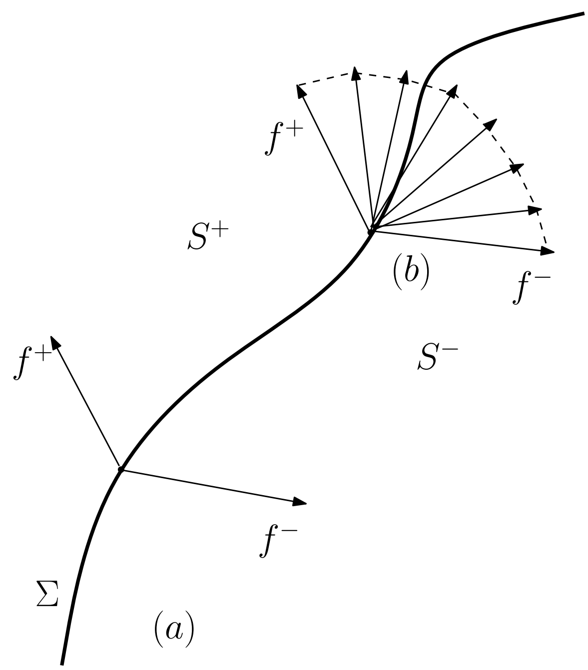


Figure 2.21: Vector fields: (a) before regularization, (b) after regularization.

Let us first define the solution of the differential inclusion (2.2) and solution in the sense of Filippov:

Definition 2.3.1 *(Solution of a Differential Inclusion) An absolutely continuous function $x : [0,\tau] \to \mathbb{R}^n$ is said to be a solution of the differential inclusion* (2.2) *if it satisfies:*

$$\dot{x}(t) \in \mathcal{F}(t, x(t)) \tag{2.8}$$

for almost all $t \in [0,\tau]$.

Definition 2.3.2 *(Solution in the Sense of Filippov) An absolutely continuous function $x : [0,\tau] \to \mathbb{R}^n$ is said to be a solution of the differential equation $\dot{x}(t) = f(t, x(t))$ in the sense of Filippov if it satisfies:*

$$\dot{x}(t) \in \mathcal{F}(t, x(t)) \tag{2.9}$$

for almost all $t \in [0, \tau]$*, where* $\mathcal{F}(t, x(t))$ *is defined in* (2.7).

Existence of solution of the differential inclusion (2.2) can be guaranteed with the notion of upper semi-continuous of the right-hand side. The following theorem is proven in [19] (Theorem 3, page 98):

Theorem 2.3.1 *(Local Existence) Assume that* $\mathcal{F}$ *is a upper semi-continuous set-valued function and that the image of* (t, x) *under* $\mathcal{F}$ *is closed, convex and bounded for all* $t \in \mathbb{R}$ *and* $x \in \mathbb{R}^n$*. Then, for each for each* $x_0 \in R^n$ *there exists a* $\tau > 0$ *and an absolutely continuous function* $x(t)$ *defined on* $[0, \tau]$*, which is a solution of the initial value problem:*

$$\dot{x}(t) \in \mathcal{F}(t, x(t)) \quad a.e.\ t \geq 0, \quad x(0) = x_0.$$

Definition 2.3.3 *The set-valued function* $\mathcal{F}$ *is called linear bounded if there exists positive constants* γ *and* c *such that:*

$$\|\mathcal{F}(t, x(t))\| \leq \gamma\|x\| + c, \quad \forall (t, x).$$

Theorem 2.3.2 *(Global Existence) Assume that* $\mathcal{F}$ *is a linear bounded and upper semi-continuous set-valued function and that the image of* (t, x) *under* $\mathcal{F}$ *is closed, convex for all* $t \in \mathbb{R}$ *and* $x \in \mathbb{R}^n$*. Then, for each for each* $x_0 \in R^n$ *there exists an absolutely continuous function* $x(t)$ *defined on* $[0, +\infty)$*, which is a solution of the initial value problem:*

$$\dot{x}(t) \in \mathcal{F}(t, x(t)), \quad x(0) = x_0. \tag{2.10}$$

In many cases, it is difficult to verify the upper-semicontinuity of $\mathcal{F}$ with respect to (t, x). Then, we may use another result proved by S. W. Seah [109] with more flexible assumptions.

Theorem 2.3.3 *Let* $\mathcal{F} : [0, +\infty[\times\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *be a set-valued function with nonempty, compact and convex values. Suppose in addition the following:*

(A) for each $t \in [0, +\infty[$*, the map* $x \mapsto \mathcal{F}(t, x)$ *is upper semi-continuous on* $\mathbb{R}^n$*,*

(B) for each $x \in \mathbb{R}^n$*, the map* $t \mapsto \mathcal{F}(t, x)$ *is measurable on* $[0, +\infty[$*;*

(C) there exist nonnegative functions f_1, f_2 *locally integrable on* $[0, +\infty[$ *such that:*

$$(\forall x \in \mathbb{R}^n, \text{ a.e. } t \in [0, +\infty[)\ (\forall w \in \mathcal{F}(t, x)) : \ \|w\| \leq f_1(t)\|x\| + f_2(t). \tag{2.11}$$

Then, for all $x_0 \in \mathbb{R}^n$*, there exists an absolutely continuous function* $x(.; x_0)$ *global solution of (2.10).*

In general, discontinuous dynamical systems do not have unique solutions. Multiplicity of solutions means that there are not enough information on the physical model to predict its behavior uniquely. The uniqueness of solution is useful for the convergence study of numerical methods for differential inclusions. There are some sufficient conditions in the literature to assure the uniqueness, such as maximal monotone, Lipschitz, one-sided Lipschitz, uniform one-sided Lipschitz, or geometrical conditions from the vector field approach [2, 19, 21, 50, 55, 77]. Let us continue with some main results on the uniqueness.

Theorem 2.3.4 *(see [77]) Consider a differential inclusion of the form:*

$$\dot{x} \in -\mathcal{B}(x). \tag{2.12}$$

If $\mathcal{B}$ is a maximal monotone set-valued function with a linear boundedness condition, then the differential inclusion has a unique solution for all initial condition.

Consider the Filippov system with the right-hand side set-valued function $\mathcal{F}$ defined in (2.7) on a domain $S \subset \mathbb{R}^n$. Without loss of generality, suppose that S is separated by a smooth surface Σ into 2 domains S^+ and S^-. Let $f(t,x)$ be smooth enough in S^- and S^+ (for example $f(t,x)$ and $\partial f/\partial x_i, i = 1, 2, \dots, n$ are continuous). Denote $f^+(t,x)$ and $f^-(t,x)$ the limiting values of f at $x \in \Sigma$ from S^+ and S^- respectively. Let f_N^+, f_N^- and h_N be the projections of f^+, f^- and h, respectively onto the normal to Σ directed from S^- to S^+ at the point x, where $h(t,x) = f^+(t,x) - f^-(t,x)$. We have the following theorems (see [50, 55]).

Theorem 2.3.5 *If at some point $x \in \Sigma$, one of the following conditions is satisfied:*

(1) $f_N^+ f_N^- > 0$,

(2) $f_N^+ < 0$ *and* $f_N^- > 0$,

then the generalized solution is unique. If

(3) $f_N^+ > 0$ *and* $f_N^- < 0$,

then a generalized solution starting from x is non-unique.

The argument of the above theorem may not be applied to the cases where $f_N^- > 0, f_N^+ = 0$, or $f_N^- = 0, f_N^+ < 0$. In these cases, we can use another result of Filippov (see [55]).

Theorem 2.3.6 *Suppose that $\Sigma \in C^2$ and $h(t,x) = f^+ - f^- \in C^1$. If for each $t \in (a,b)$ at each point $x \in \Sigma$, at least one of the inequalities $f_N^- > 0$ or $f_N^+ < 0$ (possibly different inequalities for different t and x) holds then the generalized solution is unique for $a < t < b$ in the domain S.*

Remark 2.3.1 *Note that if we have $h(t,x) < 0$ for all $t \in (a,b), x \in \Sigma$ then one of the inequalities $f_N^- > 0$ or $f_N^+ < 0$ is fulfilled for all $t \in (a,b), x \in \Sigma$.*

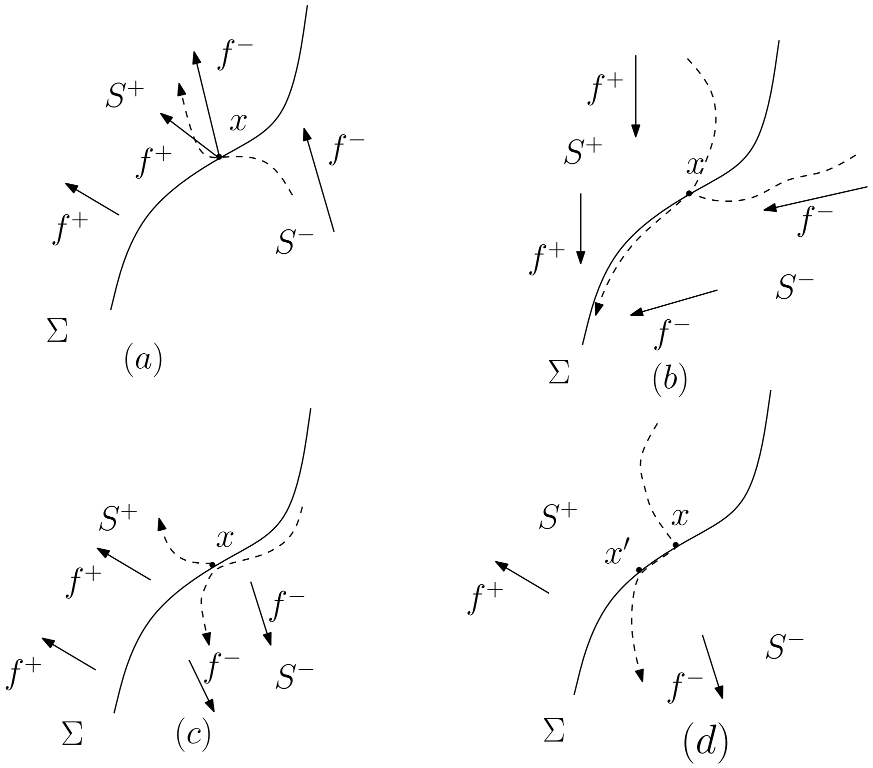


Figure 2.22: The possible cases of solutions: (a) the unique solution crosses Σ ($f_N^+ f_N^- > 0$); (b) attracting sliding mode: the unique solution once arrived on the discontinuity surface slides on it ($f_N^+ < 0$ and $f_N^- > 0$); (c) and (d) repulsing sliding mode: (c) the solution (possibly not unique) once arrived on Σ for a time instant then leaves it to any of the continuity domains ($f_N^+ > 0$ and $f_N^- < 0$), (d) the solution remains on Σ for a time interval (one of f_N^+, f_N^- is zero along a subset of Σ).

In the following part, we introduce some definitions from Lyapunov stability theory for the case of autonomous systems:

$$\dot{x} \in \mathcal{F}(x). \tag{2.13}$$

The case of non-autonomous DIs can be defined similarly. Let $\mathcal{W}$ be the set of stationary solutions:

$$\mathcal{W} := \{q \in \mathbb{R}^n : 0 \in \mathcal{F}(q)\}. \tag{2.14}$$

Without loss of generality, we assume that $0 \in \mathcal{F}(0)$ and study the stability of the origin, which is a stationary solution of (2.13).

Definition 2.3.4 *The equilibrium point $x = 0$ is said to be stable if*

$$\forall \varepsilon > 0,\ \exists \delta(\varepsilon) > 0 \textit{ such that for all } x_0 \in \mathbb{B}_{\delta(\varepsilon)} \Rightarrow \|x(t; x_0)\| \leq \varepsilon,\ \forall t \geq 0.$$

It is said unstable if it is not stable.

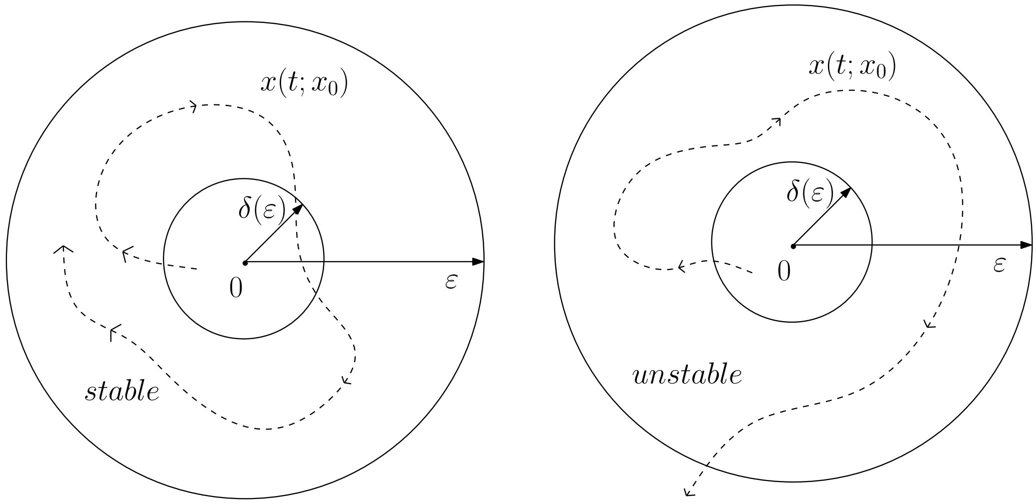


Figure 2.23: Stability of the origin.

Definition 2.3.5 *The equilibrium point $x = 0$ is said to be attractive if*

$$\exists \delta > 0 \text{ such that for all } x_0 \in \mathbb{B}_\delta \Rightarrow \lim_{t\to\infty} \|x(t;x_0)\| = 0.$$

If this is true for all $x_0 \in \mathbb{R}^n$ then $x = 0$ is said globally attractive.

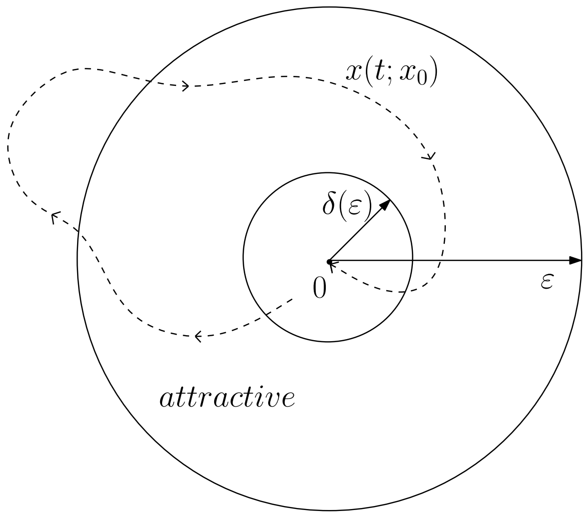


Figure 2.24: Attractivity of the origin.

Definition 2.3.6 *If the trivial equilibrium point is stable and attractive, it is called asymptotic stable; if it is stable and globally attractive, it is called globally asymptotic stable.*

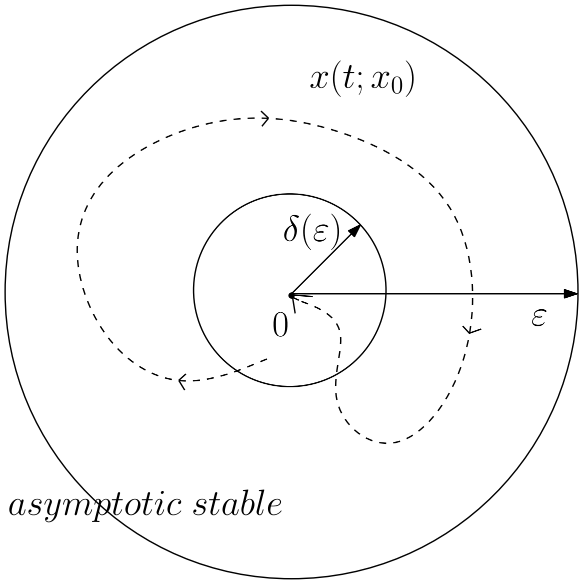


Figure 2.25: The origin is asymptotic stable.

Roughly speaking, if the effect on the motion is small for small disturbances then the undisturbed motion is call "stable". If for small disturbances, the effect on the motion is remarkable, we say that the undisturbed motion is "unstable". It is said "asymptotic stable" if for small disturbances, the effect is small and tends to disappear. If for any magnitude of disturbances, the effect is tends to disappear then the undisturbed motion is said "globally attractive".

A. M. Lyapunov introduced two distinct approaches to deal with the stability problem for both linear and nonlinear differential equations which can be also generalized for differential inclusions. The first Lyapunov's method, also called Lyapunov's indirect method, depends on finding approximate solution through linearization. More clearly, the spectrum of "linearized" operators can determine the stability of an equilibrium but it requires the smoothness of the right-hand side function. In the Lyapunov's second method, also called direct method, there is a big advantage that no such knowledge is necessary. It makes people more convenient to handle the case of non-smooth dynamical systems. The essential point in this method is proving the existence of a locally positive definite function which decreases along the trajectories of the systems. This function is called Lyapunov's function which is often chosen by the (possibly virtual) energy function.

Chapter 3

Existence and Stability Results for Electrical Circuits

3.1 Introduction

An electrical circuit is a network consisting of a closed loop that electrons from a current source or voltage flow. It combines some electrical elements such as resistors, inductors, capacitors, voltage sources, current sources, switches, diodes... The control of electric power with electronic devices has become significantly in recent decades. Although the analysis of linear circuits (including only sources; linear lumped elements such as resistors, inductors, capacitors; and linear distributed elements) is far more easy, the insight of non-linear case is still an open problem. Even circuits containing diodes ideally show non-smooth behaviors. Therefore, studying electrical circuits NSDS has an important role in practice and can be also applied in other fields. Indeed, diodes characteristics in electronics; dry friction and impact in mechanics; switching control laws in air traffic management, economic models of markets... may be modeled by non-smooth laws with similar structures. We study the electrical circuits in this chapter by means of complementarity problem and differential inclusions. Firstly, we present the ampere-volt characteristics of some electrical devices, including some non-smooth devices like diodes, diacs, silicon controller rectifiers... After that, some simple RLC circuits with diodes are analyzed by the tool of complementarity formulation. Finally, some DC-DC Buck converters are modeled by differential inclusion and their well-posedness, stability results are given. The Filippov's approach has been applied widely in mechanical switching system and it can be also used in power electronic circuits which has fruitful results. This chapter is inspired by the work of V. Acary, K. Addi, B. Brogliato [2, 5, 3, 4] and a joint work with Prof. S. Adly and Prof. D. Goeleven [11].

3.2 Ampere-Volt Characteristics of some Electrical Devices

Electrical devices can be illustrated by their corresponding ampere-volt characteristics. Each device may possess various mathematical models based on the experimental measures. For resistors, inductors and capacitors, it is known that their ampere-volt characteristics are respectively $V = Ri, V = L\frac{di}{dt}, i = C\frac{dV}{dt}$, where we use here the standard notations: V the voltage, i the current, R the resistance of resistors, L the inductance of inductors, C the capacitance of capacitors.

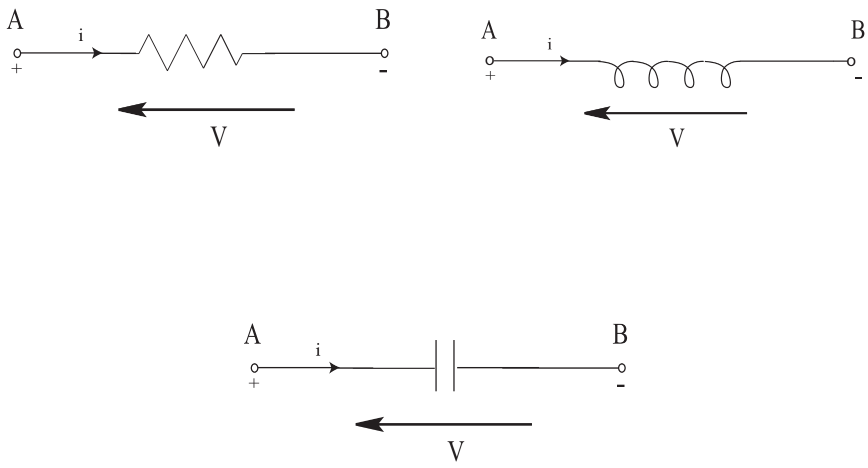


Figure 3.1: Models of a resistor, inductor and capacitor, respectively.

A diode is an electronic device which allows current to flow in one direction and blocks it in the opposite direction. Suppose it is ideal then the characteristic between the voltage $V(t)$ and the current $i(t)$ satisfies the following relation:

$$0 \leq V(t) \perp i(t) \geq 0. \tag{3.1}$$

The relation in (3.1) is called a complementarity condition, which means that both $V(t)$ and $i(t)$ are non-negative satisfying $V(t).i(t) = 0$. If the current crossing the diode is positive then its voltage is zero and vice versa.

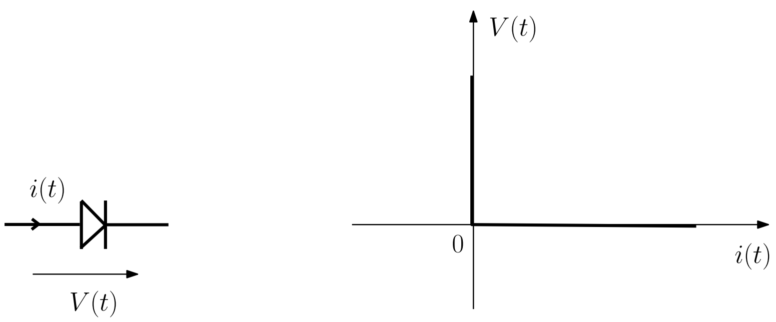


Figure 3.2: Model and characteristics of a ideal diode.

There are certainly other formulations for the diode, for example, the well-known Shockley's law defined by the equation:

$$i = I_s(e^{V/nV_T} - 1), \tag{3.2}$$

where I_s is the reverse bias saturation current, V_T is the thermal voltage, n is the ideality factor usually varying from 1 to 2. This model may have more physical meaning than the former, but for numerical purpose, the ideal model is better from both quantitative and qualitative viewpoints. We recall the subdifferential of the indicator function of $\mathbb{R}_+$(see Fig. 2.18):

$$\partial\Psi_{\mathbb{R}^+}(x) = \begin{cases} 0 & x > 0, \\ (-\infty, 0] & x = 0, \\ \emptyset & x < 0. \end{cases}$$

Therefore, the condition (3.1) can be rewritten in the form of inclusions:

$$V(t) \in -\partial\Psi_{\mathbb{R}^+}(i(t)) \Leftrightarrow i(t) \in -\partial\Psi_{\mathbb{R}^+}V(t). \tag{3.3}$$

As we will see, inclusions are preferably useful to model other more complicated devices due to its simplicity and nice tools from the theory of differential inclusions. Next, we consider an ideal Zener diode, a device that allow the current to flow not only forwardly but also reversely if the voltage is greater than a value called "Zener voltage" and denoted by $V_z > 0$. The device was named after Clarence Zener, who discovered this electrical property. The schematic symbol and the ideal characteristic between the current $i(t)$ and the voltage $v(t)$ are depicted in Fig. 3.3:

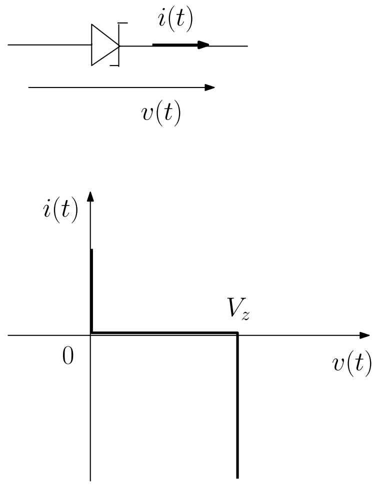


Figure 3.3: The schematic symbol and the ideal characteristic of a Zener diode.

Now, we want to express $v(t)$ as a function of $-i(t)$, more clearly to find a function $f(\cdot)$ such that $v(t) \in \partial f(-i(t))$. It is not difficult to see that:

$$f(x) = \begin{cases} V_z x & \text{if } x \geq 0, \\ 0 & \text{if } x \leq 0, \end{cases} \tag{3.4}$$

and the subdifferential of $f(\cdot)$ is:

$$\partial f(x) = \begin{cases} V_z x & \text{if } \ x > 0, \\ [0, V_z] & \text{if } x = 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{3.5}$$

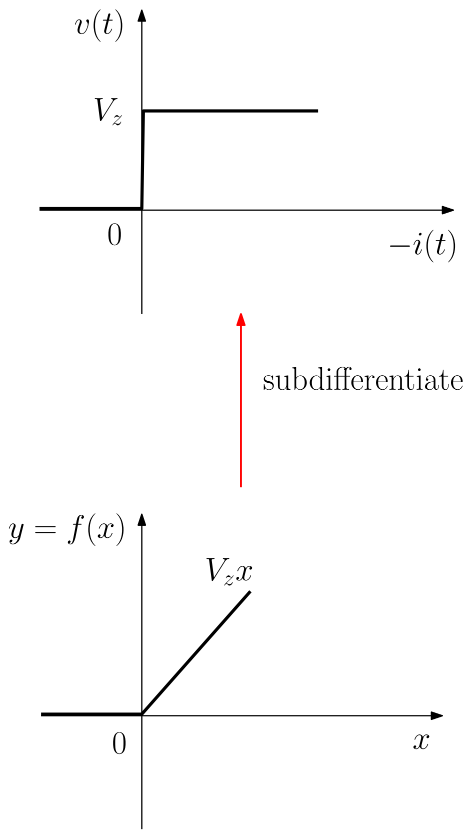


Figure 3.4: The Zener diode characteristic.

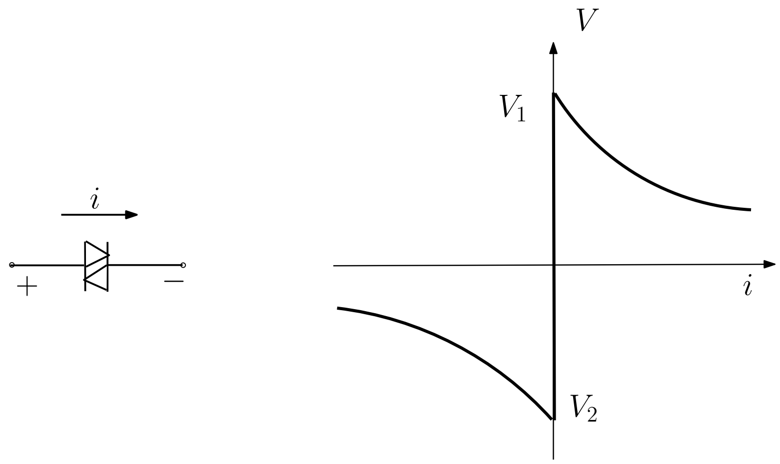


Figure 3.5: Diac model.

A diac (diode for alternating current) is a diode which allows the current to flow in both forward and reverse directions only after its voltage has reached the break-over voltage for a moment. It is also called symmetrical trigger diode because of the symmetry of their voltage-ampere characteristic. Silicon controlled rectifiers are four-layer solid state current controlling devices which are used usually in switching applications such as lamps, motors... Their symbols and characteristic models are depicted in Figure 3.6. We can see that the characteristic graphs of the electrical devices introduced above are possibly monotone and non-monotone set-valued mappings.

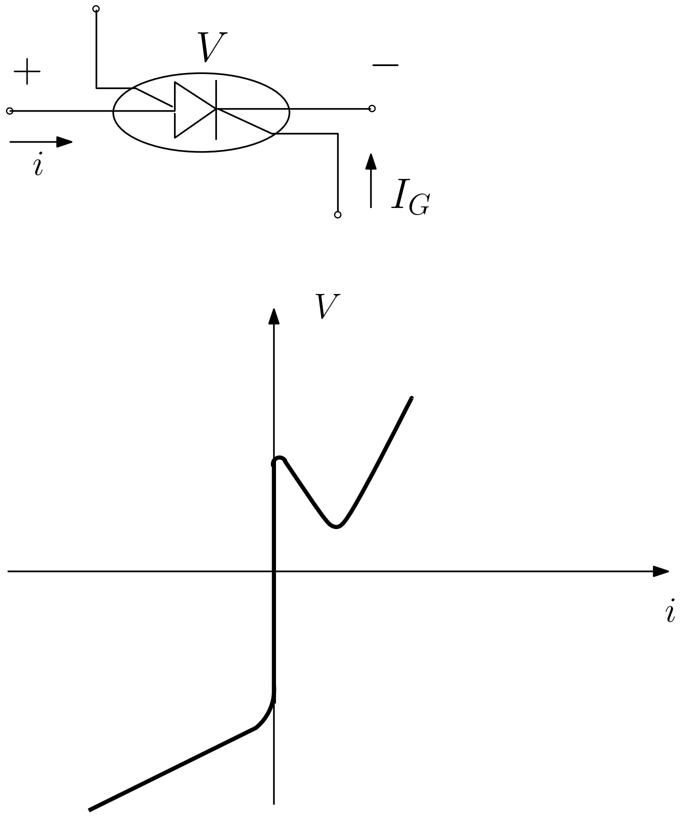


Figure 3.6: Silicon controlled rectifier model.

Let us introduce the Moreau-Panagiotopoulos' approach to give a mathematical treatment for such possibly set-valued characteristics. This work has been initially studied by Moreau [92] for the case of monotone set-valued graphs and developed by Panagiotopoulos [97] for the general case which includes the monotone and non-monotone graphs. This approach has been used widely not only by mathematicians to study theoretically but also by engineers to model highly nonlinear phenomena in mechanics such as friction, unilateral contact, fracture... For an electrical device, we may write its voltage-ampere their characteristic in the form $V \in \mathcal{F}(i)$ where $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ is a set-valued function. One of the steps in Moreau-Panagiotopoulos' work is finding a possibly discontinuous function $g \in L^{\infty}_{loc}(\mathbb{R};\mathbb{R})$ such that both left and right limits of g exist everywhere and for all $t \in \mathbb{R}$, one has:

$$\mathcal{F}(t) = [\min\{g(t^-), g(t^+)\}, \max\{g(t^-), g(t^+)\}].$$

If we define a new function f by:

$$f(t) = \int_0^t g(s)ds,$$

then f is locally-Lipschitz due to the fact that $g \in L^{\infty}_{loc}(\mathbb{R};\mathbb{R})$. Then, there is a known result [45, 62] that the Clarke's subdifferential of f at t can be computed by:

$$\partial f(t) = [\min\{g(t^-), g(t^+)\}, \max\{g(t^-), g(t^+)\}].$$

Consequently, the initial problem may be reduced to the subdifferential form: $V \in \partial f(i)$ to enjoy nice properties of Clarke's subdifferential and power tools of unilateral analysis.

3.3 Analysis of some simple RLC Circuits with Diodes

In this section, we analyze some simple RLC circuits with diodes by tool of complementarity formulation. This approach makes more convenient in numerical simulation. Let us first introduce some basic definition and result about linear complementarity problem (LCP).

Definition 3.3.1 *Let a vector $q \in R^m$ and a matrix $M \in \mathbb{R}^{m\times m}$ be given. A LCP (q, M) is a problem that consists of finding a vector $\lambda \in \mathbb{R}^m$ such that:*

$$0 \leq \lambda \perp M\lambda + q \geq 0. \tag{3.6}$$

The following theorem is a fundamental result of complementarity theory:

Theorem 3.3.1 *The LCP (q, M) has a unique solution λ^* for all $q \in R^m$ if only if M is a P-matrix. In this case the solution λ^* is a piecewise linear function of q.*

Remark 3.3.1 *A matrix is called P-matrix if all its principal minors are positive. It is obvious that a positive definite matrix is a P-matrix. A symmetric P-matrix is a positive definite matrix.*

A *linear complementarity system* (LCS) is a special dynamical system defined by a linear ordinary differential equation (ODE) involving an algebraic variable that is required to be a solution of a standard (LCP):

$$\begin{cases} \dot{x} = Ax + B\lambda, \\ y = Cx + D\lambda, \\ 0 \leq y \perp \lambda \geq 0. \end{cases}$$

Using a backward Euler scheme to evaluate the time derivative $\dot{x}$, we have as follows:

$$\begin{cases} \frac{x_{k+1}-x_k}{h} = Ax_{k+1} + B\lambda_{k+1}, \\ y_{k+1} = Cx_{k+1} + D\lambda_{k+1}, \\ 0 \leq y_{k+1} \perp \lambda_{k+1} \geq 0, \end{cases}$$

which implies by a straightforward substitution:

$$0 \leq \lambda_{k+1} \perp C(I - hA)^{-1}x_k + (hC(I - hA)^{-1}B + D)\lambda_{k+1} \geq 0. \tag{3.7}$$

For examples, we consider three following non-smooth electrical circuits :

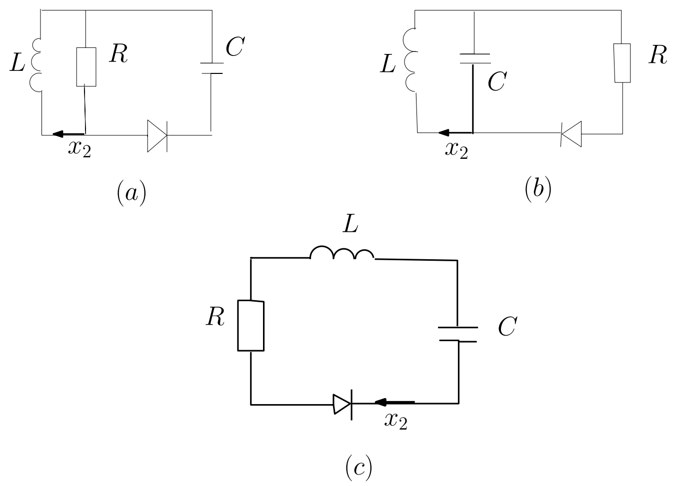


Figure 3.7: RLC circuits with an ideal diode.

Let us recall two important Kirchoff's laws:

1. **Kirchhoff′s voltage law** : The algebraic sum of the voltages between successive nodes in all meshes in the circuit are zero.
2. **Kirchhoff′s current law**: The algebraic sum of the currents in all branches which converge to a common node equal zero.

Taking into account the Kirchoff's laws as well as constitutive relations of resistors, inductors and capacitors, we obtain the dynamical equations of 3 circuits depicted in Fig. 3.7:

$$(a)\begin{cases} \dot{x}_1(t) = x_2(t) - \frac{1}{RC}x_1(t) - \frac{\lambda(t)}{R}, \\ \dot{x}_2(t) = -\frac{1}{LC} - \frac{\lambda(t)}{L}, \\ 0 \le \lambda(t) \perp w(t) = \frac{\lambda(t)}{R} + \frac{1}{RC}x_1(t) - x_2(t) \ge 0. \end{cases} \tag{3.8}$$

$$(b)\begin{cases} \dot{x}_1(t) = -x_2(t) + \lambda(t) \\ \dot{x}_2(t) = \frac{1}{LC}x_1(t) \\ 0 \le \lambda(t) \perp w(t) = \frac{1}{C}x_1(t) + R\lambda(t) \ge 0 \end{cases} \tag{3.9}$$

$$(c)\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\frac{R}{L}x_2(t) - \frac{1}{LC}x_1(t) - \frac{\lambda(t)}{L} \\ 0 \le \lambda(t) \perp w(t) = -x_2(t) \ge 0 \end{cases} \tag{3.10}$$

where $x_1(t), x_2(t)$ denote the current through the inductors and the charge on the capacitors, respectively. The complementarity conditions play an essential role in the dynamics.

The variable λ may be interpreted as a Lagrange multiplier.

Circuit (a): It is easy to see that there is a unique solution $\lambda(t)$ to the LCP in (3.8) given by:

$$\lambda(t) = 0 \qquad \text{if } \frac{1}{RC}x_1(t) - x_2(t) \geq 0, \tag{3.11}$$

$$\lambda(t) = -\frac{1}{C}x_1(t) + Rx_2(t) > 0 \qquad \text{if } \frac{1}{RC}x_1(t) - x_2(t) < 0. \tag{3.12}$$

Therefore, the system (3.8) can be rewritten in the form of piecewise linear system as follows:

$$\left[\begin{array}{ll} \left\{\begin{array}{l} \dot{x}_1(t) = x_2(t) - \frac{1}{RC}x_1(t) \\ \dot{x}_2(t) = -\frac{1}{LC}x_1(t) \end{array}\right. & \text{if } -\frac{1}{C}x_1(t) + Rx_2(t) < 0, \\ & \\ \left\{\begin{array}{l} \dot{x}_1(t) = 0 \\ \dot{x}_2(t) = -\frac{R}{L}x_2(t). \end{array}\right. & \text{if } -\frac{1}{C}x_1(t) + Rx_2(t) \geq 0. \end{array}\right. \tag{3.13}$$

or

$$\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{B}.\text{Proj}_{\mathbb{R}_+}\left[-\frac{1}{C}x_1(t) + Rx_2(t)\right], \tag{3.14}$$

where matrices $\mathbf{A}$ and $\mathbf{B}$ can be easily found.

The projection operator in the right-hand side of (3.14) is a Lipschitz-continuous single valued function. Thus we can conclude that this LCS has a global unique and differentiable solution. Numerical simulation of circuit (a) can be done by any standard ODE solvers (forward or backward Euler, mid-point, trapezoidal, Runge-Kutta of order 4...). Here, we use a standard Runge-Kutta of order 4 to simulate. The result is depicted in Figure 3.8 with $x_1(0) = 1, x_2(0) = -1, R = 10, L = 1, C = \frac{1}{(2\pi)^2}$ and the time step $h = 0.005$:

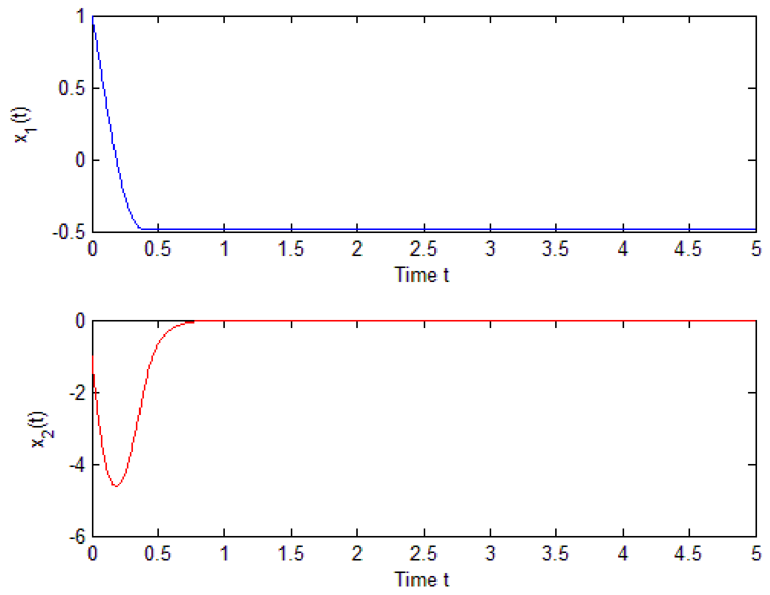


Figure 3.8: Simulation of circuit (a).

Circuit (b): It can be analyzed in the same way done for circuit (a). Simulation of this case is given in Figure 3.9:

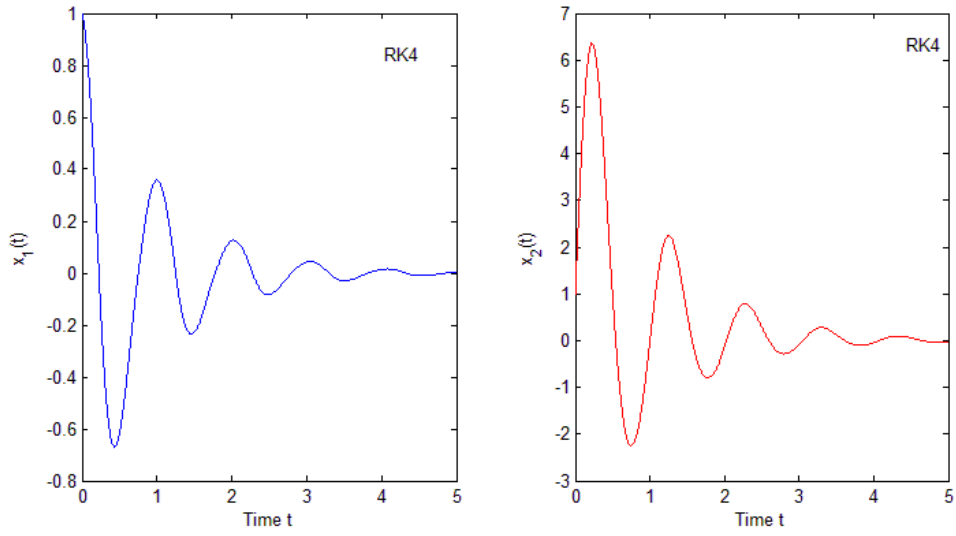


Figure 3.9: Simulation of circuit (b) with the initial condition $x_1(0) = 1, x_2(0) = 1, R = 10, L = 1, C = \frac{1}{(2\pi)^2}$ and $h = 0.005$.

Circuit (c): This time, we cannot compute directly $\lambda(t)$ from the complementarity relations:

$$0 \leq \lambda(t) \perp w(t) = -x_2(t) \geq 0. \tag{3.15}$$

There is no LCP here (indeed, we have a zero matrix M). However, we can apply the backward Euler discretization for this system:

$$\begin{cases} x_{1,k+1} - x_{1,k} = hx_{2,k+1}, \\ x_{2,k+1} - x_{2,k} = -h\frac{R}{L}x_{2,k+1} - \frac{h}{LC}x_{1,k+1} - \frac{h}{L}\lambda_{k+1}, \\ 0 \leq \lambda_{k+1} \perp -x_{2,k+1} \geq 0, \end{cases} \tag{3.16}$$

where x_k is the value of the approximation solution at time t_k of a equidistant grid $t_0 < t_1 < \ldots < t_N = T, h = \frac{T-t_0}{N} = t_k - t_{k-1}$.

Let $a_h = 1 + h\frac{R}{L} + \frac{h^2}{LC}$. After some simple computations, system (3.16) can be rewritten as:

$$\begin{cases} x_{1,k+1} - x_{1,k} = hx_{2,k+1}, \\ x_{2,k+1} = a_h^{-1}\left(x_{2,k} - \frac{h}{LC}x_{1,k} - \frac{h}{L}\lambda_{k+1}\right), \\ 0 \leq \lambda_{k+1} \perp -a_h^{-1}\left(x_{2,k} - \frac{h}{LC}x_{1,k}\right) + a_h^{-1}\frac{h}{L}\lambda_{k+1} \geq 0. \end{cases} \tag{3.17}$$

We can solve the LCP in (3.17) easily:

$$h\lambda_{k+1} = 0 \qquad \text{if } \frac{h}{LC}x_{1,k} - x_{2,k} > 0, \tag{3.18}$$

$$h\lambda_{k+1} = L\left(x_{2,k} - \frac{h}{LC}x_{1,k}\right) \qquad \text{if } \frac{h}{LC}x_{1,k} - x_{2,k} \leq 0. \tag{3.19}$$

Substituting these values into (3.17), we obtain:

$$x_{2,k+1} = \begin{cases} a_h^{-1}(x_{2,k} - \frac{h}{LC}x_{1,k}) & \text{if } \frac{h}{LC}x_{1,k} - x_{2,k} > 0, \\ 0 & \text{if } \frac{h}{LC}x_{1,k} - x_{2,k} \leq 0. \end{cases} \tag{3.20}$$

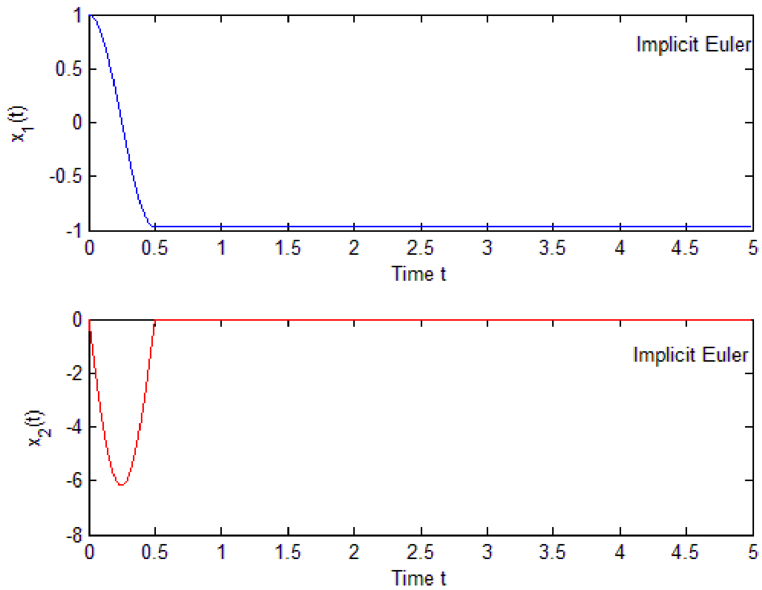


Figure 3.10: Simulation of circuit (c) with the initial condition $x_1(0) = 1, x_2(0) = 0, R = 0.1, L = 1, C = \frac{1}{(2\pi)^2}$ and $h = 0.001$.

Electrical Circuit with Ideal Zener Diode: We consider the dynamics of the **circuit** (c) but replace the ideal diode by a Zener ideal diode. Then, we obtain:

$$\left\{ \begin{array}{l} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) + \frac{R}{L} x_2(t) + \frac{1}{LC} x_1(t) = \frac{1}{L} v(t), \\ v(t) \in \partial f(-i(t)). \end{array} \right. \tag{3.21}$$

Note that the condition $v(t) \in \partial f(-i(t))$ is equivalent to:

$$\left\{ \begin{array}{l} 0 \leq \lambda_1(t) \perp -i(t) + |i(t)| \geq 0, \\ 0 \leq \lambda_2(t) \perp i(t) + |i(t)| \geq 0, \\ \lambda_1(t) + \lambda_2(t) = V_z, \\ v(t) = \lambda_2(t), \end{array} \right. \tag{3.22}$$

where λ_1, λ_2 are two slack variables (or multipliers). Denote u^+, u^- the positive part and the negative part of u, respectively:

$$u^+ = \frac{1}{2}(u + |u|) \geq 0, \tag{3.23}$$

$$u^- = \frac{1}{2}(u - |u|) \leq 0. \tag{3.24}$$

$$\tag{3.25}$$

Then, the system (3.21) can be rewritten as:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) + \frac{R}{L}x_2(t) + \frac{1}{LC}x_1(t) = \frac{1}{L}\lambda_2(t), \\ x_2^+(t) = x_2(t) - x_2^-(t), \\ \lambda_1(t) = V_z - \lambda_2(t), \\ 0 \leq \begin{pmatrix} x_2^+(t) \\ \lambda_1(t) \end{pmatrix} \perp \begin{pmatrix} \lambda_2(t) \\ -x_2^-(t) \end{pmatrix} \geq 0. \end{cases} \tag{3.26}$$

Using the backward Euler scheme, we have:

$$\begin{cases} x_{1,k+1} - x_{1,k} = hx_{2,k+1}, \\ x_{2,k+1} - x_{2,k} + h\frac{R}{L}x_{2,k+1} + \frac{h}{LC}x_{1,k+1} = \frac{h}{L}\lambda_{2,k+1}, \\ x_{2,k+1}^+ = x_{2,k+1} - x_{2,k+1}^-, \\ \lambda_{1,k+1} = V_z - \lambda_{2,k+1}, \\ 0 \leq \begin{pmatrix} x_{2,k+1}^+ \\ \lambda_{1,k+1} \end{pmatrix} \perp \begin{pmatrix} \lambda_{2,k+1} \\ -x_{2,k+1}^- \end{pmatrix} \geq 0. \end{cases} \tag{3.27}$$

Let $a_h = \left(1 + h\frac{R}{L} + \frac{h^2}{LC}\right)^{-1}$ and $b = a_h\left(x_{2,k} - \frac{h}{LC}x_{1,k}\right)$. After some manual computations, we obtain:

$$\begin{cases} x_{1,k+1} - x_{1,k} = hx_{2,k+1}, \\ x_{2,k+1} - b = a_h\frac{h}{L}\lambda_{2,k+1}, \\ x_{2,k+1}^+ = x_{2,k+1} - x_{2,k+1}^-, \\ \lambda_{1,k+1} = V_z - \lambda_{2,k+1}, \\ 0 \leq \begin{pmatrix} x_{2,k+1}^+ \\ \lambda_{1,k+1} \end{pmatrix} \perp \begin{pmatrix} \lambda_{2,k+1} \\ -x_{2,k+1}^- \end{pmatrix} \geq 0. \end{cases} \tag{3.28}$$

Then, the value of $\lambda_{2,k+1}$ and $x_{2,k+1}$ can be computed at each step by the following LCP:

$$\begin{cases} w = \begin{pmatrix} \frac{a_h h}{L} & 1 \\ -1 & 0 \end{pmatrix} z + \begin{pmatrix} b \\ V_z \end{pmatrix}, \\ 0 \leq w \perp z \geq 0. \end{cases} \tag{3.29}$$

The numerical simulation result of the system is given as follows:

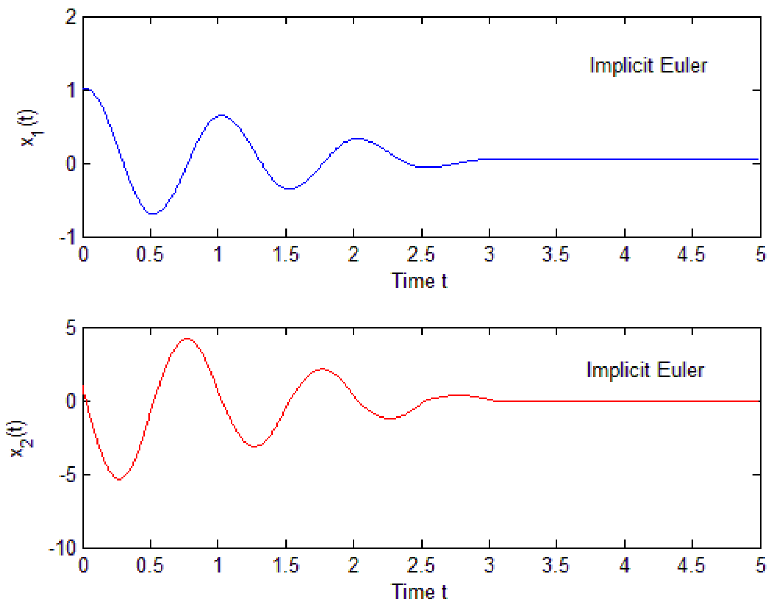


Figure 3.11: Simulation of the RLC circuit with a Zener diode with the initial condition $x_1(0) = 1, x_2(0) = 1, R = 0.1, L = 1, C = \frac{1}{(2\pi)^2}$ and $h = 0.005$.

3.4 Stability Analysis of a DC-DC Buck Converter

In this section, we study stability properties of a power DC-DC Buck converter which is known to show non-smooth behavior. It is switching dynamical system characterized by discrete switching events. We study its behavior by means of NSDS, or more clearly, differential inclusions, which can capture the important properties of the system. Thanks to the powerful tools from non-smooth and variational analysis [19, 93, 102, 103, 111], the existence of a solution, stability and asymptotical stability results for our model can be obtained. Finally, we have a discussion about the numerical simulation for the system trajectories.

3.4.1 DC-DC Buck Converter Model

Let us consider the circuit in Figure 3.12. This second order circuit is known as a useful power DC-DC Buck converter. Experimental set-ups describing the nonlinear and nonsmooth dynamics can be found in [15]. We will propose a mathematical formulation of the nonsmooth model by using tools from nonsmooth and variational analysis.

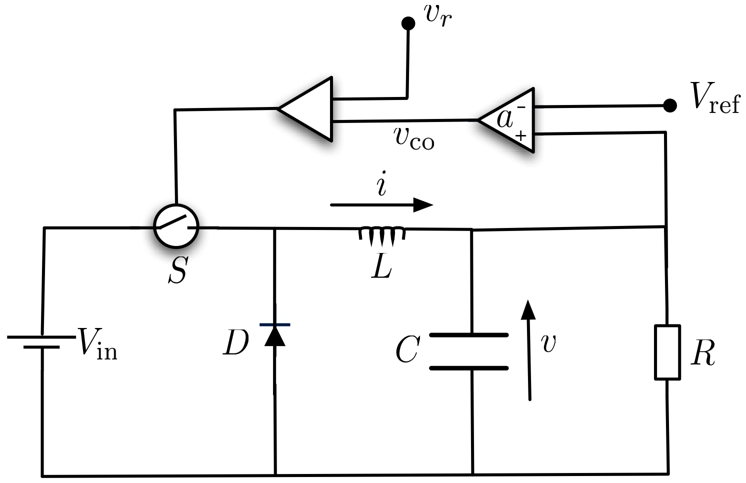


Figure 3.12: DC-DC Buck Converter

The parameters of the circuit are the resistance $R > 0$, the capacitance $C > 0$ and the inductance $L > 0$. The voltage v_r is assumed to be described by a given function on $\mathbb{R}_+$. We also denote by $a > 0$ the gain of the amplifier, V_{ref} the reference voltage, and $V_{in} > 0$ the input voltage. The voltage v of the capacitor is applied to the positive pole of the amplifier and the reference voltage to the negative pole. The output voltage, also called control voltage v_{co} is thus given by:

$$v_{co}(t) = a(v(t) - V_{ref}).$$

It is then compared with the given voltage v_r to decide the switch S ON or OFF. The switch is ON if and only if $v_{co} < v_r$ or equivalently, $v < V_{ref} + \frac{1}{a}v_r$.

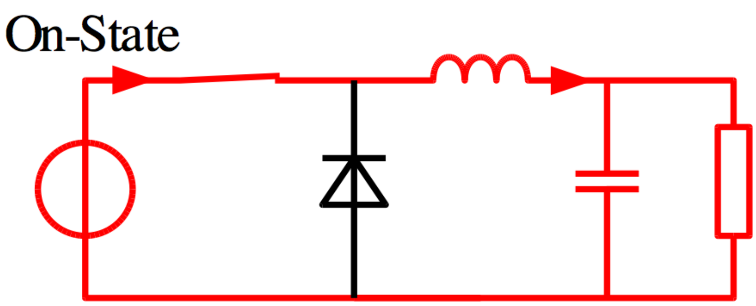


Figure 3.13: The switch is ON: $V_L + V_C = V_{in}$.

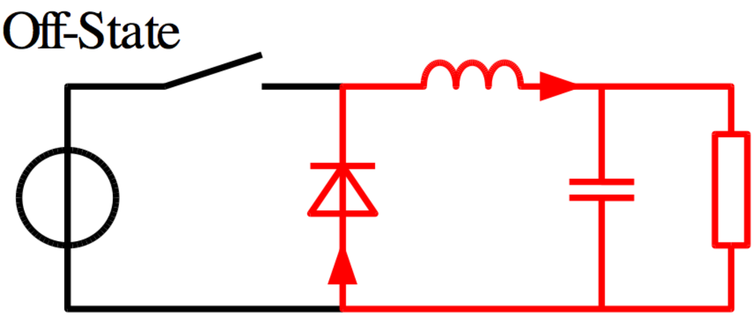


Figure 3.14: The switch is OFF: $V_L + V_C = 0$.

Using Kirchhoff's circuit laws, we have:

$$V_L + v = \begin{cases} V_{in} & \text{if } S \text{ is ON,} \\ 0 & \text{if } S \text{ is OFF.} \end{cases}$$

Note that: $V_L = L\frac{dI_L}{dt}$ and $I_L = I_R + I_C = \frac{v}{R} + C\frac{dv}{dt}$. Therefore, we have $V_L = LCv'' + \frac{L}{R}v'$ and the system becomes:

$$LCv'' + \frac{L}{R}v' + v = \begin{cases} V_{in} & \text{if } S \text{ is ON,} \\ 0 & \text{if } S \text{ is OFF.} \end{cases}$$

Therefore, the Buck converter can be described by the second order differential inclusion:

$$LCv''(t) + \frac{L}{R}v'(t) + v(t) \in V_{in}u(t, v(t)) \tag{3.30}$$

where $u(t, v)$ is defined by:

$$u(t, v) = \frac{1}{2}\Big(\text{Sign}(V_r(t) - v) + 1\Big)$$

with

$$V_r(t) = V_{ref} + \frac{1}{a}v_r(t).$$

The function $\text{Sign} : \mathbb{R} \rightrightarrows \mathbb{R},\ x \mapsto \text{Sign}(x)$ is the set-valued function defined by:

$$\text{Sign}(x) = \begin{cases} -1 & \text{if} \quad x < 0, \\ [-1, +1] & \text{if} \quad x = 0, \\ +1 & \text{if} \quad x > 0. \end{cases}$$

We have:

$$u(t, v) = \begin{cases} 1 & \text{if} \quad v < V_r(t), \\ [0, 1] & \text{if} \quad v = V_r(t), \\ 0 & \text{if} \quad v > V_r(t). \end{cases} \tag{3.31}$$

Remark 3.4.1 *The control law u in (3.30) considered in [15] is in fact given by the discontinuous function:*

$$u(t, v(t)) = \begin{cases} 0 & if \quad v(t) > V_r(t), \\ \\ 1 & if \quad v(t) < V_r(t). \end{cases} \tag{3.32}$$

Indeed, we have used the Filippov's approach (see [55]) that provides a good mathematical tool to study the differential equation with discontinuous right-hand side. It consists indeed to imbed the discontinuous function (3.32) into the set-valued map (3.31) so as to enjoy enough regularity properties to develop a rigorous mathematical theory.

Remark 3.4.2 *The DC-DC Buck converter may be controlled with a classical Pulse Width Modulation (see [15]) with a ramp known as sawtooth function:*

$$v_r(t) = V_L + (V_U - V_L)(\frac{t}{T} - [\frac{t}{T}]) \tag{3.33}$$

where for $z \in \mathbb{R}$, $[z]$ stands for the integer part of z, V_L and V_U are the lower and upper voltages of the ramp and $T > 0$ is its period.

Let us now consider the second order differential inclusion:

$$LCv''(t) + \frac{L}{R}v'(t) + v(t) \in \frac{V_{in}}{2} + \frac{V_{in}}{2}\text{Sign}(V_r(t) - v(t)).$$

We may set $x_2 = v$ and $x_1 = v'$ to obtain the first order differential inclusion:

$$x_1'(t) \in -\frac{1}{RC}x_1(t) - \frac{1}{LC}x_2(t) + \frac{V_{in}}{2LC} + \frac{V_{in}}{2LC}\text{Sign}(V_r(t) - x_2(t))$$

and then the first order differential inclusion system:

$$\begin{pmatrix} x_1'(t) \\ \\ x_2'(t) \end{pmatrix} \in \begin{pmatrix} -\frac{1}{RC} & -\frac{1}{LC} \\ \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ \\ x_2(t) \end{pmatrix} + \begin{pmatrix} \frac{V_{in}}{2LC} \\ \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{V_{in}}{2LC}\text{Sign}(V_r(t) - x_2(t)) \\ \\ 0 \end{pmatrix}.$$

Using the notations:

$$x = \begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix}, \ A = \begin{pmatrix} -\frac{1}{RC} & -\frac{1}{LC} \\ \\ 1 & 0 \end{pmatrix}, \ b = \begin{pmatrix} \frac{V_{in}}{2LC} \\ \\ 0 \end{pmatrix},$$

and

$$F(t, x) = \begin{pmatrix} \frac{V_{in}}{2LC}\text{Sign}(V_r(t) - x_2) \\ \\ 0 \end{pmatrix}, \tag{3.34}$$

our problem reduces to the system:

$$x'(t) \in Ax(t) + b + F(t, x(t)), \tag{3.35}$$

with some given initial condition:

$$x(0) = x_0, \tag{3.36}$$

where $x_0 \in \mathbb{R}^2$.

3.4.2 Existence of trajectories in a DC-DC Buck converter

In this part, we discuss the existence of solutions of (3.30) through the dynamical system (3.35)-(3.36). The following result is obtained as a direct consequence of Theorem 2.3.3.

Theorem 3.4.1 *Let $A \in \mathbb{R}^{2\times 2}$, $b \in \mathbb{R}^2$ and $F : [0, +\infty[\times\mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ as defined in (3.34). We suppose that for all $h \in \mathbb{R}$, the set-valued mapping $t \to \mathrm{Sign}(V_r(t) - h)$ is measurable. Then, for any $x_0 \in \mathbb{R}^2$, there exists an absolutely continuous function $x(.; x_0)$ such that:*

$$\begin{cases} x(0; x_0) = x_0, \\ \\ x'(t; x_0) \in Ax(t; x_0) + b + F(t, x(t; x_0)), \text{ a.e. } t \in [0, +\infty[. \end{cases}$$

Proof. We prove that all assumptions of Theorem 2.3.3 are satisfied by the mapping $G : [0, +\infty[\rightrightarrows \mathbb{R}^2; (t, x) \mapsto G(t, x)$ defined by $G(t, x) = Ax + b + F(t, x)$. It is clear that, for all $t \in [0, +\infty[$ and for all $x \in \mathbb{R}^2$, the set $G(t, x)$ is nonempty, compact and convex.

Let $t \in \mathbb{R}_+$ be given, the single-valued function $x \mapsto V_r(t) - x_2$ is continuous and thus upper semi-continuous on $\mathbb{R}^2$. The set-valued function $z \mapsto \mathrm{Sign}(z)$ is upper semi-continuous on $\mathbb{R}$. It results that the function $x \mapsto \mathrm{Sign}(V_r(t) - x_2)$ is upper semi-continuous on $\mathbb{R}^2$ as the composite of two upper semi-continuous functions. The single-valued function $x \mapsto Ax + b$ is continuous and thus upper semi-continuous on $\mathbb{R}^2$. Therefore, the mapping $x \mapsto G(t, x)$ is upper semi-continuous on $\mathbb{R}^2$ as the sum of two upper semi-continuous mappings on $\mathbb{R}^2$.

Let $x \in \mathbb{R}^2$ be given. The set-valued function $t \mapsto \mathrm{Sign}(V_r(t) - x_2)$ is assumed measurable and so is the mapping $t \mapsto F(t, x)$ and consequently the mapping $t \mapsto G(t, x)$.

Let $t \in [0, +\infty[$, $x \in \mathbb{R}^2$ and $w \in G(t, x)$ be given. We may write $w = Ax + b + w_F$ with $w_F \in F(t, x)$. It is clear that $\|w_F\| \leq \frac{|V_{in}|}{2LC}$ and thus $\|w\| \leq \|A\|\|x\| + \|b\| + \frac{|V_{in}|}{2LC}$. The function G is thus linear bounded and the existence result follows from Theorem 2.3.3. ■

Example 3.4.1 *If the function $t \mapsto v_r(t)$ is continuous then the function $t \mapsto V_r(t) - x_2$ is continuous and thus upper semi-continuous. The set-valued mapping $t \mapsto \mathrm{Sign}(V_r(t) - x_2)$ is thus upper semi-continuous as the composite of two upper semi-continuous functions. Corollary III.3 in [43] ensures that $t \mapsto \mathrm{Sign}(V_r(t) - x_2)$ is measurable.*

Example 3.4.2 *Let us consider a Pulse Width Modulation control with a ramp defined in (3.33) and recall that $V_r(t) = V_{ref} + \frac{1}{a}v_r(t)$. Let $x \in \mathbb{R}^2$ be given. We have:*

$$\mathrm{Sign}(V_r(t) - x_2) = \begin{cases} +1 & \textit{if} \quad x_2 < V_r(t), \\ \\ [-1, +1] & \textit{if} \quad x_2 = V_r(t), \\ \\ -1 & \textit{if} \quad x_2 > V_r(t), \end{cases}$$

with

$$V_r(t) = V_{ref} + \frac{V_L}{a} + \frac{(V_U - V_L)}{a}\left(\frac{t}{T} - [\frac{t}{T}]\right).$$

Let $x \in \mathbb{R}^2$ *be given. If* $x_2 \geq V_{ref} + \frac{V_U}{a}$ *then for all* $t \geq 0$ *we have* $x_2 > V_r(t)$ *and thus:*

$$(\forall t \geq 0) : \mathrm{Sign}(V_r(t) - x_2) = -1.$$

It results in particular that the mapping $t \mapsto \mathrm{Sign}(V_r(t) - x_2)$ *is measurable.*

If $x_2 < V_{ref} + \frac{V_L}{a}$ *then for all* $t \geq 0$ *we have* $x_2 < V_r(t)$ *and:*

$$(\forall t \geq 0) : \mathrm{Sign}(V_r(t) - x_2) = 1.$$

In this case, the mapping $t \mapsto \mathrm{Sign}(V_r(t) - x_2)$ *is also measurable.*

Suppose now that $V_{ref} + \frac{V_L}{a} \leq x_2 < V_{ref} + \frac{V_U}{a}$*. We denote by* $t^* = t^*(x_2)$ *the unique solution in the interval* $[0, T]$ *of the equation* $x_2 = V_r(t)$*, i.e.,*

$$t^* := (x_2 - V_{ref} - \frac{V_L}{a})\frac{Ta}{V_U - V_L}.$$

Let $i \in \mathbb{N}, i \geq 1$ *be given. We have:*

$$\mathrm{Sign}(V_r(t) - x_2) = \begin{cases} -1 & \textit{if} \quad iT \leq t < t^* + iT, \\ [-1, +1] & \textit{if} \quad t = t^* + iT, \\ +1 & \textit{if} \quad t^* + iT < t < (1+i)T. \end{cases}$$

To check that the latter set-valued mapping is measurable, it is necessary and sufficient (see Theorem 2.2.3) to verify that for all $z \in \mathbb{R}$*, the single-valued function* $t \mapsto d(z, \mathrm{Sign}(V_r(t) - x_2))$ *is measurable. Let* $z \in \mathbb{R}$ *be given. If* $z > 1$ *then:*

$$d(z, \mathrm{Sign}(V_r(t) - x_2)) = \begin{cases} |z| + 1 & \textit{if} \quad iT \leq t < t^* + iT, \\ |z| - 1 & \textit{if} \quad t^* + iT \leq t < (1+i)T. \end{cases}$$

If $z < -1$ *then:*

$$d(z, \mathrm{Sign}(V_r(t) - x_2)) = \begin{cases} |z| - 1 & \textit{if} \quad iT \leq t \leq t^* + iT, \\ |z| + 1 & \textit{if} \quad t^* + iT < t < (1+i)T. \end{cases}$$

If $0 \leq z \leq 1$ *then:*

$$d(z, \mathrm{Sign}(V_r(t) - x_2)) = \begin{cases} |z| + 1 & \textit{if} \quad iT \leq t < t^* + iT, \\ 0 & \textit{if} \quad t = t^* + iT, \\ 1 - |z| & \textit{if} \quad t^* + iT < t < (1+i)T, \end{cases}$$

and if $-1 \leq z < 0$ *then*

$$d(z, \mathrm{Sign}(V_r(t) - x_2)) = \begin{cases} 1 - |z| & \textit{if} \quad iT \leq t < t^* + iT, \\ 0 & \textit{if} \quad t = t^* + iT, \\ |z| + 1 & \textit{if} \quad t^* + iT < t < (1+i)T. \end{cases}$$

In each case, the single-valued function $t \mapsto d(z, \mathrm{Sign}(V_r(t) - x_2))$ *is measurable as a step-function.*

Remark 3.4.3 *In general, we do not have the uniqueness of solutions of the differential inclusion (3.30). One usually check some kind of conditions like hypo-monotone, maximal monotone, one-sided Lipschitz... to ensure the uniqueness which can not be satisfied for our system, unfortunately. Indeed, we may consider the example in* (3.43) *with initial data* $V_0 > 0$, *then* $V(t) \equiv V_0$ *and* $V(t) = V_0 e^{-\gamma t}$ *are two different solutions.*

3.4.3 Stationary solution in a DC-DC Buck converter model

In the following two sections, we suppose that the voltage v_r is described by a constant function on $\mathbb{R}_+$. The Buck converter model is then autonomous and we consider the problem of equilibrium for solution of (3.30). So here, let us suppose that:

$$(\forall t \geq 0): \; v_r(t) = v_r$$

where $v_r \in \mathbb{R}$. Then $(\forall t \geq 0): \; V_r(t) = V_{ref} + \frac{1}{a} v_r$. We set:

$$V_r = V_{ref} + \frac{1}{a} v_r. \tag{3.37}$$

The differential inclusion:

$$LCv''(t) + \frac{L}{R} v'(t) + v(t) \in \frac{V_{in}}{2} + \frac{V_{in}}{2} \mathrm{Sign}(V_r - v(t))$$

may be rewritten as:

$$v''(t) + \frac{1}{RC} v'(t) + \frac{1}{LC} v(t) \in \frac{V_{in}}{2LC} - \frac{1}{LC} \partial \varphi(v(t)) \tag{3.38}$$

where φ is the convex and Lipschitz continuous function:

$$(\forall z \in \mathbb{R}): \; \varphi(z) = \frac{V_{in}}{2} \mid z - V_r \mid .$$

A stationary solution $\overline{v}$ is then given as solution of the differential inclusion:

$$\overline{v} + \partial \varphi(\overline{v}) \ni \frac{V_{in}}{2}. \tag{3.39}$$

It is well known that the mapping $(I + \partial \varphi)^{-1}$ is a well-defined single-valued operator and $\overline{v}$ is then uniquely defined by:

$$\overline{v} = (I + \partial \varphi)^{-1} (\frac{V_{in}}{2}).$$

It is also well known that $(I + \partial \varphi)^{-1}(.) = \mathrm{argmin}_{z \in \mathbb{R}} \frac{1}{2} |z - .|^2 + \varphi(z)$ and thus:

$$\overline{v} = \mathrm{argmin}_{z \in \mathbb{R}} \frac{1}{2} |z - \frac{V_{in}}{2}|^2 + \frac{V_{in}}{2} \mid z - V_r \mid .$$

We have the following proposition.

Proposition 3.4.1 *The differential inclusion (3.38) admits a unique stationary solution given by:*

$$\overline{v} = \begin{cases} 0 & \text{if} \quad V_r < 0, \\ V_r & \text{if} \quad V_r \in [0, V_{in}], \\ V_{in} & \text{if} \quad V_r > V_{in}. \end{cases} \tag{3.40}$$

Proof. We know that the stationary solution of (3.38) is given as the unique solution of the inclusion (3.39). It suffices to show that $\overline{v}$ defined in (3.40) solves (3.39). We have:

$$\partial\varphi(\overline{v}) = \begin{cases} -\frac{V_{in}}{2} & \text{if} \quad \overline{v} < V_r, \\ [-\frac{V_{in}}{2}, +\frac{V_{in}}{2}] & \text{if} \quad \overline{v} = V_r, \\ +\frac{V_{in}}{2} & \text{if} \quad \overline{v} > V_r. \end{cases}$$

If $V_r < 0$ then $\overline{v} = 0$ satisfies (3.39) since $0 + \partial\varphi(0) = \frac{V_{in}}{2}$. If $V_r > V_{in}$ then $\overline{v} = V_{in}$ solves (3.39) since $V_{in} + \partial\varphi(V_{in}) = V_{in} - \frac{V_{in}}{2} = \frac{V_{in}}{2}$. Finally, if $V_r \in [0, V_{in}]$ then $V_r - \frac{V_{in}}{2} \in [-\frac{V_{in}}{2}, +\frac{V_{in}}{2}]$ and thus V_r solves (3.39). ■

3.4.4 Asymptotic mathematical properties of trajectories in a DC-DC Buck converter

In this section, we also suppose that $(\forall t \geq 0): \ v_r(t) = v_r$. Our aim is to prove that the equilibrium solution given in Proposition 3.4.1 is globally attractive in the sense that for any $v_0, w_0 \in \mathbb{R}$ and any solution $v(.; v_0, w_0)$ of (3.30) satisfying the initial conditions $v(0; v_0, w_0) = v_0$ and $v'(0; v_0, w_0) = w_0$ then:

$$\lim_{t \to +\infty} v(t; v_0, w_0) = \overline{v}.$$

Let us set:

$$V = v - \overline{v}$$

to rewrite our model as:

$$V"(t) + \frac{1}{RC}V'(t) + \frac{1}{LC}V(t) \in \frac{V_{in}}{2LC} - \frac{\overline{v}}{LC} - \frac{1}{LC}\partial\varphi(\overline{v} + V(t)).$$

Or equivalently:

$$V"(t) + \alpha V'(t) \in -\partial\Phi(V(t)) \tag{3.41}$$

with $\alpha = \frac{1}{RC}$ and

$$(\forall V \in \mathbb{R}): \ \Phi(V) = \frac{1}{2LC}|V|^2 + (\frac{\overline{v}}{LC} - \frac{V_{in}}{2LC})V + \frac{1}{LC}\varphi(\overline{v} + V).$$

We note that the trivial solution 0 is the unique stationary solution of (3.41) (see Proposition 3.4.1).

Remark 3.4.4 *i) The mapping Φ is convex and locally Lipschitz continuous.*

ii) We have:

$$\Phi(0) = \frac{1}{LC}\varphi(\overline{v}) = \frac{V_{in}}{2LC}|\overline{v} - V_r|.$$

Proposition 3.4.2 *We have:*

$$\text{Argmin } \Phi = \{0\}.$$

Proof. A classical result in convex analysis (see e.g. [102]) ensures that:

$$(\forall Z \in \mathbb{R}): \ \Phi(\overline{V}) \leq \Phi(Z) \ \Leftrightarrow \ 0 \in \partial\Phi(\overline{V}).$$

Here

$$\partial\Phi(\overline{V}) = \frac{1}{LC}\overline{V} + \frac{\overline{v}}{LC} - \frac{V_{in}}{2LC} + \frac{1}{LC}\partial\varphi(\overline{v} + \overline{V}).$$

Thus

$$0 \in \partial\Phi(\overline{V}) \Leftrightarrow 0 \in \frac{1}{LC}(\overline{V} + \overline{v}) - \frac{V_{in}}{2LC} + \frac{1}{LC}\partial\varphi(\overline{V} + \overline{v})$$

$$\Leftrightarrow \frac{V_{in}}{2} \in (\overline{V} + \overline{v}) + \partial\varphi(\overline{V} + \overline{v}) \Leftrightarrow \overline{V} + \overline{v} = (I + \partial\varphi)^{-1}(\frac{V_{in}}{2}) = \overline{v}$$

$$\Leftrightarrow \overline{V} = 0.$$

■

Let $V : [0, +\infty[\to \mathbb{R}$ be a solution of the differential inclusion (3.41) and satisfying the initial conditions $V(0) = V_0$ and $V'(0) = W_0$. We set:

$$E(t) = \frac{1}{2}|V'(t)|^2 + \Phi(V(t)).$$

Theorem 3.4.2 *For all* $0 \leq t_1 \leq t_2$*, we have:*

$$E(t_2) - E(t_1) = -\alpha \int_{t_1}^{t_2} |V'(s)|^2 ds.$$

Proof. There exists a function ξ such that:

$$\xi(t) \in \partial\Phi(V(t)), \text{ a.e. } t \geq 0$$

and

$$V"(t) + \alpha V'(t) = -\xi(t), \text{ a.e. } t \geq 0.$$

The function $t \mapsto V(t)$ is absolutely continuous and the function $t \mapsto \Phi(V(t))$ is also absolutely continuous as the composite of the locally Lipschitz continuous mapping Φ and the absolutely continuous mapping V. Thus $\Phi \circ V$ is almost everywhere differentiable. We may write for a.e. $t \geq 0$ and for all $\varepsilon > 0$:

$$\frac{\Phi(V(t+\varepsilon)) - \Phi(V(t))}{\varepsilon} \geq (\frac{V(t+\varepsilon) - V(t)}{\varepsilon})\xi(t)$$

and

$$\frac{\Phi(V(t-\varepsilon)) - \Phi(V(t))}{-\varepsilon} \leq (\frac{V(t-\varepsilon) - V(t)}{-\varepsilon})\xi(t).$$

Taking the limit as $\varepsilon \to 0^+$ in these two formulas, we get:

$$\frac{d}{dt}\Phi(V(t)) \geq V'(t)\xi(t)$$

and

$$\frac{d}{dt}\Phi(V(t)) \leq V'(t)\xi(t).$$

Hence

$$\frac{d}{dt}\Phi(V(t)) = V'(t)\xi(t).$$

We also have:

$$V"(t)V'(t) + \alpha(V'(t))^2 = -\xi(t)V'(t), \text{ a.e. } t \geq 0$$

and thus:

$$\frac{1}{2}\frac{d}{ds}|V'(s)|^2 + \frac{d}{ds}\Phi(V(s)) = -\alpha|V'(s)|^2, \text{ a.e. } s \geq 0.$$

It results that for all $0 \leq t_1 \leq t_2$:

$$\frac{1}{2}\int_{t_1}^{t_2}\frac{d}{ds}|V'(s)|^2 ds + \int_{t_1}^{t_2}\frac{d}{ds}\Phi(V(s))ds = -\alpha\int_{t_1}^{t_2}|V'(s)|^2 ds.$$

Thus

$$\frac{1}{2}|V'(t_2)|^2 - \frac{1}{2}|V'(t_1)|^2 + \Phi(V(t_2)) - \Phi(V(t_1)) = -\alpha\int_{t_1}^{t_2}|V'(s)|^2 ds.$$

■

Remark 3.4.5 *It results from Theorem 3.4.2 that the mapping $t \mapsto E(t)$ is a nonincreasing function on $\mathbb{R}_+$.*

Proposition 3.4.3 *The functions $t \mapsto V(t)$ and $t \mapsto \Phi(V(t))$ are bounded on $[0, +\infty[$.*

Proof. Note that $\Phi(V(t)) \geq \Phi(0)$ (Proposition 3.4.2). From Theorem 3.4.2, we deduce:

$$(\forall t \geq 0): \ \Phi(V(t)) = \Phi(V(0)) + \frac{1}{2}|V'(0)|^2 - \frac{1}{2}|V'(t)|^2 - \alpha\int_0^t |V'(s)|^2 ds \leq$$

$$\leq \Phi(V_0) + \frac{1}{2}|W_0|^2.$$

Let us set $C_0 = \Phi(V_0) + \frac{1}{2}|W_0|^2$. We have:

$$(\forall t \geq 0): \ \frac{1}{2LC}|V(t)|^2 + (\frac{\overline{v}}{LC} - \frac{V_{in}}{2LC})V(t) + \frac{V_{in}}{2LC}|\overline{v} + V(t) - V_r| \leq C_0.$$

Then

$$(\forall t \geq 0): \ \frac{1}{2LC}|V(t)|^2 - |\frac{\overline{v}}{LC} - \frac{V_{in}}{2LC}||V(t)| - \frac{V_{in}}{2LC}|V(t)| \leq C_0 + \frac{V_{in}}{2LC}|\overline{v} - V_r|.$$

Setting $k_1 = \frac{1}{2LC}$, $k_2 = |\frac{\overline{v}}{LC} - \frac{V_{in}}{2LC}| + \frac{V_{in}}{2LC}$ and $k_0 = C_0 + \frac{V_{in}}{2LC}|\overline{v} - V_r|$, then we have:

$$(\forall t \geq 0): \ k_1|V(t)|^2 - k_2|V(t)| \leq k_0.$$

We claim that there exists $M > 0$ such that:

$$(\forall t \geq 0): \ |V(t)| \leq M.$$

Indeed, if we suppose the contrary, we may asserts that:

$$(\forall i \in \mathbb{N}, i \neq 0) \ (\exists t_i \geq 0): \ |V(t_i)| > i$$

so that:

$$\lim_{i \to +\infty} |V(t_i)| = +\infty.$$

However:

$$(\forall i \in \mathbb{N}, i \neq 0): k_1 |V(t_i)|^2 - k_2 |V(t_i)| \leq k_0$$

and thus:

$$(\forall i \in \mathbb{N}, i \neq 0): k_1 |V(t_i)| - k_2 \leq \frac{k_0}{|V(t_i)|}.$$

Taking the limit as $i \to +\infty$, we obtain the contradiction $+\infty \leq 0$. ■

Proposition 3.4.4 *We have:*

$$V' \in L^\infty(\mathbb{R}_+) \cap L^2(\mathbb{R}_+).$$

Proof. We have:

$$(\forall t \geq 0): \ \frac{1}{2}|V'(t)|^2 = -\Phi(V(t)) + \frac{1}{2}|W_0|^2 + \Phi(V_0) - \alpha \int_0^t |V'(s)|^2 ds.$$

Since $\forall t \geq 0, \Phi(V(t)) \geq \Phi(0)$, we obtain:

$$(\forall t \geq 0): \ \frac{1}{2}|V'(t)|^2 \leq \frac{1}{2}|W_0|^2 + \Phi(V_0) - \Phi(0) < +\infty.$$

It results that $V' \in L^\infty(\mathbb{R}_+)$. We have also:

$$\alpha \int_0^t |V'(s)|^2 ds = -\Phi(V(t)) + \frac{1}{2}|W_0|^2 + \Phi(V_0) - \frac{1}{2}|V'(t)|^2$$

$$\leq \Phi(V_0) + \frac{1}{2}|W_0|^2 - \Phi(0) < +\infty$$

and thus $V' \in L^2(\mathbb{R}_+)$. ■

Proposition 3.4.5 *We have:*

$$\lim_{t \to +\infty} E(t) = \min \ \Phi,$$

$$\lim_{t \to +\infty} \Phi(V(t)) = \min \ \Phi = \Phi(0), \ \ \frac{1}{2}|V'(t)|^2 \leq E(t) - \Phi(0),$$

and

$$\lim_{t \to +\infty} |V'(t)|^2 = 0.$$

Proof. Let $k : \mathbb{R}_+ \to \mathbb{R}$ be the function defined by:

$$(\forall t \geq 0): \; k(t) = V'(t)V(t) + \frac{\alpha}{2}|V(t)|^2.$$

We have for a.e. $t \geq 0$:

$$k'(t) = V"(t)V(t) + |V'(t)|^2 + \alpha V(t)V'(t)$$

and thus for all $0 \leq t_1 \leq t_2$, we get:

$$k(t_2) - k(t_1) = \int_{t_1}^{t_2} (V"(s) + \alpha V'(s))V(s)ds + \int_{t_1}^{t_2} |V'(s)|^2 ds.$$

We know that:

$$V"(s) + \alpha V'(s) \in -\partial\Phi(V(s))$$

and thus:

$$\Phi(W) - \Phi(V(s)) \geq -(V"(s) + \alpha V'(s))(W - V(s)), \; \forall \, W \in \mathbb{R}.$$

Setting $W = 0$, we get:

$$(V"(s) + \alpha V'(s))V(s) \leq \Phi(0) - \Phi(V(s)).$$

Therefore:

$$k(t_2) - k(t_1) \leq \int_{t_1}^{t_2} (\Phi(0) - \Phi(V(s)))ds + \int_{t_1}^{t_2} |V'(s)|^2 ds.$$

Recalling that:

$$\Phi(V(s)) = E(s) - \frac{1}{2}|V'(s)|^2$$

we may also write:

$$k(t_2) - k(t_1) + \int_{t_1}^{t_2} (E(s) - \Phi(0))ds \leq \frac{3}{2}\int_{t_1}^{t_2} |V'(s)|^2 ds.$$

We know that the function $t \mapsto E(t)$ is nonincreasing (see Remark 3.4.5) on $\mathbb{R}_+$. The function $t \mapsto E(t)$ is also bounded from below on $\mathbb{R}_+$ since $(\forall t \in \mathbb{R}_+): \; E(t) = \Phi(V(t)) + \frac{1}{2}|V'(t)|^2 \geq \Phi(V(t))$ and the mapping $t \mapsto \Phi(V(t))$ is bounded from below on $\mathbb{R}_+$ (see Proposition 3.4.3). It results that:

$$\lim_{t \to +\infty} E(t) = E$$

with

$$E = \inf_{s \in R_+} E(s) \in \mathbb{R}.$$

On the other hand, we have:

$$k(t) + \int_0^t (E(s) - \Phi(0))ds \leq k(0) + \frac{3}{2}\int_0^t |V'(s)|^2 ds$$

and

$$k(t) \geq -|V'(t)||V(t)| + \frac{\alpha}{2}|V(t)|^2.$$

Recalling that $V' \in L^\infty(\mathbb{R}_+)$, we set $|V'|_\infty := \sup_{s\in\mathbb{R}_+} |V'(s)|$ to write:

$$k(t) \geq -|V'|_\infty|V(t)| + \frac{\alpha}{2}|V(t)|^2$$

$$\geq -\frac{1}{\alpha}|V'|^2_\infty + \frac{\alpha}{4}|V(t)|^2.$$

The last inequality follows from the fact that:

$$(\forall Z \in \mathbb{R}) : \frac{\alpha}{4}Z^2 - |V'|_\infty Z + \frac{|V'|^2_\infty}{\alpha} \geq 0.$$

Thus:

$$k(t) \geq -\frac{1}{\alpha}|V'|^2_\infty$$

and consequently:

$$\int_0^t (E(s) - \Phi(0))ds \leq k(0) + \frac{1}{\alpha}|V'|^2_\infty + \frac{3}{2}\int_0^t |V'(s)|^2 ds$$

From

$$\Phi(V(s)) + \frac{1}{2}|V'(s)|^2 = E(s)$$

we have that $(\forall z \in \mathbb{R}) :\ E(z) \geq \Phi(V(z)) \geq \min \Phi = \Phi(0)$ and hence $E \geq \Phi(0)$. Let us now check that $E \leq \Phi(0)$. Indeed, suppose on the contrary that $E > \Phi(0)$. Since

$$\int_0^t (E(s) - \Phi(0))ds \geq \int_0^t (E - \Phi(0))ds = (E - \Phi(0))t,$$

we get:

$$(E - \Phi(0))t \leq \int_0^t (E(s) - \Phi(0))ds \leq k(0) + \frac{1}{\alpha}|V'|^2_\infty + \frac{3}{2}\int_0^t |V'(s)|^2 ds$$

and then:

$$(E - \Phi(0))t \leq k(0) + \frac{1}{\alpha}|V'|^2_\infty + \frac{3}{2}\int_0^{+\infty} |V'(s)|^2 ds.$$

Note that $K := k(0) + \frac{1}{\alpha}|V'|^2_\infty + \frac{3}{2}\int_0^{+\infty} |V'(s)|^2 ds < +\infty$ since $V' \in L^\infty(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. Taking the limit as $t \to +\infty$, we get the contradiction $+\infty \leq K$.

Thus

$$E = \Phi(0) = \min\ \Phi.$$

We have:

$$E(t) - \Phi(0) \geq \Phi(V(t)) - \Phi(0) \geq 0$$

and taking the limit as $t \to +\infty$, we obtain: $0 \geq \lim_{t\to+\infty} \Phi(V(t)) - \Phi(0) \geq 0$ and thus:

$$\lim_{t\to+\infty} \Phi(V(t)) = \Phi(0).$$

We also have:

$$E(t) - \Phi(0) = \Phi(V(t)) + \frac{1}{2}|V'(t)|^2 - \Phi(0) \geq \frac{1}{2}|V'(t)|^2 \geq 0.$$

Thus $E(t) - \Phi(0) \geq \frac{1}{2}|V'(t)|^2 \geq 0$ and taking the limit as $t \to +\infty$, we get:

$$\lim_{t \to +\infty} |V'(t)|^2 = 0.$$

■

Let us here recall the Opial lemma.

Lemma 3.4.1 *(Opial) Let H be a Hilbert space and $v : [0, +\infty[\to \mathbb{R}$ be a function such that there exists a nonempty set $S \subset H$ satisfying (i) for all $t_n \to +\infty$ with $v(t_n) \to v_\infty$ weakly in H, we have $v_\infty \in S$ and (ii) for all $z \in S$, $\lim_{t \to +\infty} |v(t) - z|$ exists. Then $v(t)$ weakly converges as $t \to +\infty$ to some element v_∞ of S.*

Theorem 3.4.3 *The trivial solution of* (3.41) *is globally attractive in the sense that if $V(.; V_0, W_0)$ is a solution of* (3.41) *satisfying given initial conditions $V(0; V_0, W_0) = V_0$ and $V'(0; V_0, W_0) = W_0$ then:*

$$\lim_{t \to +\infty} V(t; V_0, W_0) = 0.$$

Proof. We apply Opial lemma with $H = \mathbb{R}$ and $S = \{0\}$ to the function $V \equiv V(.; V_0; W_0)$. Let us first check assumption (i) of Opial lemma. Taking $V(t_n) \to V_\infty$, we have by continuity of the mapping Φ and Proposition 3.4.5 that:

$$\Phi(V_\infty) = \lim_{t \to +\infty} \Phi(V(t_n)) = \min \Phi.$$

Using Proposition 3.4.2, we obtain $V_\infty \in \text{Argmin}\, \Phi = \{0\}$.
Let us now check assumption (ii) of Opial lemma, i.e. that the limit $\lim_{t \to +\infty} |V(t)|$ exists. Using the notations of the proof of Proposition 3.4.5, we know that for $0 \leq t_1 \leq t_2$:

$$k(t_2) - k(t_1) \leq \frac{3}{2} \int_{t_1}^{t_2} |V'(s)|^2 ds - \int_{t_1}^{t_2} (E(s) - \Phi(0)) ds \leq \frac{3}{2} \int_{t_1}^{t_2} |V'(s)|^2 ds$$

since $E(s) \geq \Phi(V(s)) \geq \Phi(0)$ for all $s \geq 0$. It results that the mapping $t \to k(t) - \frac{3}{2} \int_0^t |V'(s)|^2 ds$ is nonincreasing. Moreover:

$$(\forall t \in \mathbb{R}_+) : \quad k(t) - \frac{3}{2} \int_0^t |V'(s)|^2 ds \geq -\frac{1}{\alpha} |V'|_\infty - \frac{3}{2} \int_0^\infty |V'(s)|^2 ds$$

and thus the mapping:

$$t \mapsto k(t) - \frac{3}{2} \int_0^t |V'(s)|^2 ds,$$

is also bounded from below since $-\frac{1}{\alpha}|V'|_\infty - \frac{3}{2} \int_0^\infty |V'(s)|^2 ds \in \mathbb{R}$. It results that the limit:

$$\lim_{t \to +\infty} \left(k(t) - \frac{3}{2} \int_0^t |V'(s)|^2 ds \right) = l \in \mathbb{R}$$

where

$$l = \inf_{t\in\mathbb{R}_+} \{k(t) - \frac{3}{2}\int_0^t |V'(s)|^2 ds\}.$$

Then

$$\lim_{t\to+\infty} k(t) = l + \frac{3}{2}\int_0^\infty |V'(s)|^2 ds.$$

On the other hand, the mapping $t \mapsto V(t)$ is bounded (see Proposition 3.4.3) and $\lim_{t\to+\infty} V'(t) = 0$ (see Proposition 3.4.5). Thus:

$$\lim_{t\to+\infty} V'(t)V(t) = 0.$$

Then

$$\lim_{t\to+\infty} |V(t)| = \lim_{t\to+\infty} \sqrt{\frac{2}{\alpha}(k(t) - V'(t)V(t))} = \sqrt{\frac{2}{\alpha}(l + \frac{3}{2}\int_0^\infty |V'(s)|^2 ds)}.$$

Thus the limit $\lim_{t\to+\infty} |V(t)|$ exists. Opial lemma ensures that:

$$\lim_{t\to+\infty} V(t) = 0.$$

■

Theorem 3.4.4 *Consider the case of* $V_r \in [0, V_{in}]$. *Then, the trivial solution of* (3.41) *is stable in the sense of Lyapunov, i.e, for each* $\varepsilon > 0$, *there exists an* $\eta > 0$ *such that if* $\sqrt{V_0^2 + W_0^2} < \eta$ *then*

$$\sqrt{|V(t, V_0, W_0)|^2 + |V'(t, V_0, W_0)|^2} < \varepsilon,$$

for all $t \geq 0$.

Proof. When $V_r \in [0, V_{in}]$, we have $\bar{v} = V_r$. Then for all $V \in \mathbb{R}$:

$$\Phi(V) = \frac{1}{2LC}|V|^2 + (\frac{\overline{v}}{LC} - \frac{V_{in}}{2LC})V + \frac{V_{in}}{2LC}|V|.$$

Set $x_1 = V, x_2 = V'$ and $x = (x_1\ x_2)^T$, we reduce (3.41) into the first order system:

$$\dot{x}(t) \in F(x(t)) \tag{3.42}$$

where

$$F(x(t)) = \begin{pmatrix} x_2(t) \\ -\alpha x_2(t) - \partial\Phi(x_1(t)) \end{pmatrix}.$$

Consider the Lyapunov function $W: \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$W(x) = \frac{1}{2}x_2^2 + \Phi(x_1).$$

From Remark 3.4.5, the derivative of W along the trajectories of (3.42) is non-positive. Note that $\Phi(0) = 0$ and $\{0\} = \text{Argmin}\ \Phi$, in particular, Φ is a locally positive definite function and so is W. Then, there exist $h > 0$ and a strictly increasing function $\alpha(\cdot) \in C(\mathbb{R}^+;\mathbb{R})$ with $\alpha(0) = 0$ such that:

$$W(x) \geq \alpha(\|x\|) \text{ for all } x \in \mathbb{B}_h.$$

Without loss of generality, let $0 < \varepsilon < h$ and let $c = \alpha(\varepsilon)$. Since W is locally positive definite, there exists a $\eta > 0$ such that $\mathbb{B}_\eta \subset \Omega_c^\circ = \{x \in \mathbb{R}^2 : W(x) < c\}$. Let $\delta = \min\{\varepsilon, \eta\}$. Take $x_0 \in \mathbb{B}_\delta$ and $x(t; 0, x_0)$ is a solution of (3.42) satisfying the initial condition $x(0) = x_0$. Suppose that there exists $t_1 \geq 0$ such that $\|x(t_1; 0, x_0)\| \geq \varepsilon$. Since $x(\cdot; t_0, x_0)$ is continuous, we may find some t^* satisfying: $\|x(t^*; 0, x_0)\| = \varepsilon$. Then

$$W(x(t^*; 0, x_0)) \geq \alpha(\|x(t^*; 0, x_0)\|) = \alpha(\varepsilon).$$

On the other hand, W is decreasing along the trajectory, we have:

$$W(x(t^*; 0, x_0)) \leq W(x_0) < c = \alpha(\varepsilon).$$

Our proof is finished by the contradiction. ■

Remark 3.4.6 *In general, we do not have the finite-time stability of the system. For example, consider the case of $V_r = 0$. Then $\bar{v} = V_r = 0$. Let $\alpha = \frac{1}{RC}, \beta = \frac{1}{LC}$, our system becomes:*

$$V''(t) + \alpha V'(t) + \beta V(t) \in \frac{\beta V_{in}}{2} - \frac{\beta V_{in}}{2} \mathrm{Sign}(V(t)). \tag{3.43}$$

Suppose we have the condition that $\alpha^2 \geq 4\beta$, or equivalently, $L \geq 4R^2C$ and let γ be a root of the equation $y^2 - \alpha y + \beta = 0$. Then $V(t) = e^{-\gamma t}$ is a solution of (3.43) *and $V(t)$ does not converge to* 0 *in a finite time.*

3.4.5 Numerical Simulation

Consider the differential inclusion (3.41), or equivalently, the differential inclusion (3.42) with $R = 10, L = 1, C = 1/(2\pi)^2, V_0 = 3, W_0 = 1$. Note that:

$$\partial\Phi(V(t)) = \frac{V(t)}{LC} + (\frac{\bar{v}}{LC} - \frac{V_{in}}{2LC}) + \frac{V_{in}}{2LC}\mathrm{Sign}(\bar{v} + V(t) - V_r).$$

If $V_r < 0$ or $V_r > V_{in}$ then $\bar{v} \neq V_r$. Therefore, the sign of $\bar{v} + V(t) - V_r$ does not change when $V(t)$ is very close to 0. In these cases, we can use an explicit Euler time-discretzation for (3.42) :

$$\begin{cases} x_{1,k+1} - x_{1,k} = hx_{2,k}, \\ x_{2,k+1} \in x_{2,k} + h(-\alpha x_{2,k} - \partial\Phi(x_{1,k+1})). \end{cases}$$

where x_k is the value of an approximate step function $x^n(\cdot)$, at time t_k of a grid $t_0 < t_1 < \ldots < t_n, h = \frac{t_n - t_0}{n}$ (Figs. 5.11 and 3.16).

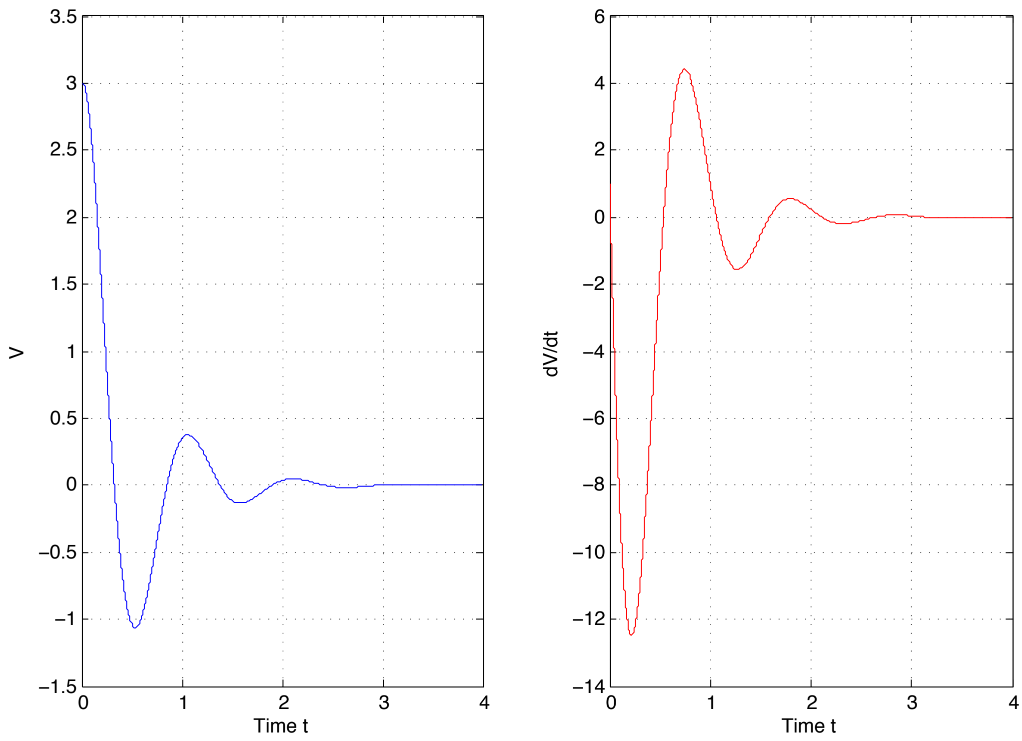


Figure 3.15: Numerical simulation for (3.41) with $V_{in} = 7, V_r = -3$ $(V_r < 0)$.

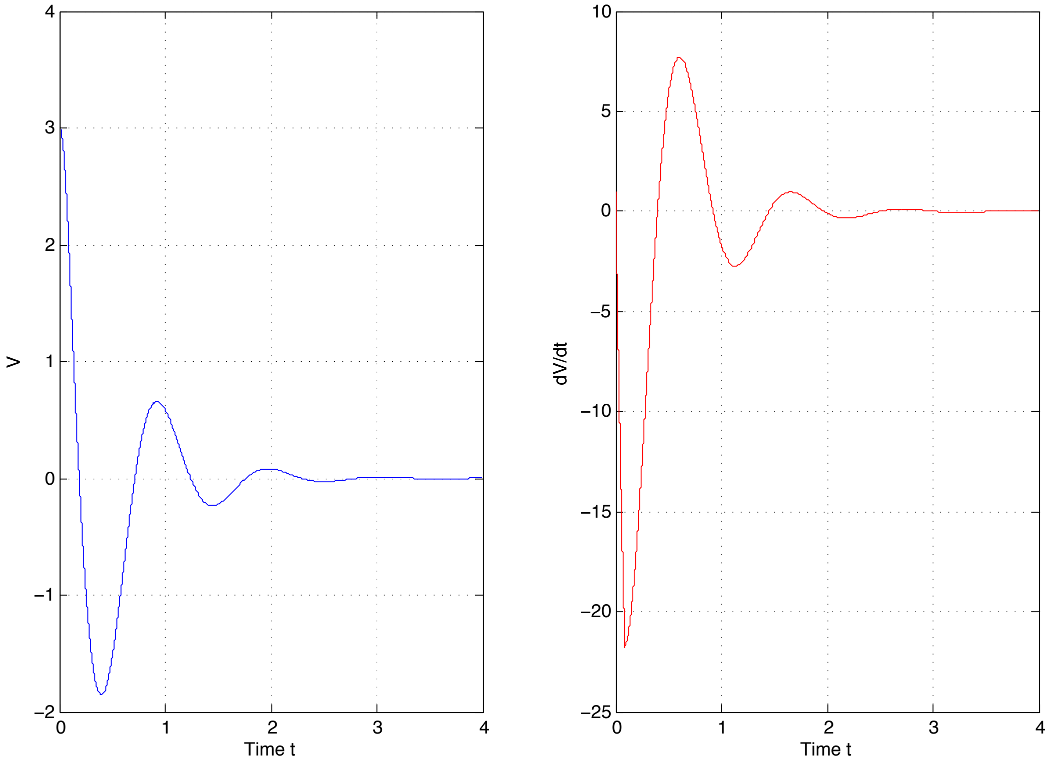


Figure 3.16: Numerical simulation for (3.41) with $V_{in} = 5, V_r = 7$ $(V_r > V_{in})$.

In the case of $V_r \in [0, V_{in}]$, we have $\bar{v} = V_r$ and:

$$\partial\Phi(V(t)) = \frac{V(t)}{LC} + (\frac{\bar{v}}{LC} - \frac{V_{in}}{2LC}) + \frac{V_{in}}{2LC}\mathrm{Sign}(V(t)).$$

When $V(t) \to 0$, the values of $\mathrm{Sign}(V(t))$ change remarkably. It is the reason why if we use an explicit Euler scheme here, we may receive unwanted effects like chattering effects around 0. Therefore, in this case we use an implicit Euler scheme:

$$\left\{ \begin{array}{c} x_{1,k+1} - x_{1,k} = hx_{2,k+1}, \\ x_{2,k+1} \in x_{2,k} + h(-\alpha x_{2,k} - \partial\Phi(x_{1,k+1})). \end{array} \right. \tag{3.44}$$

Then, we obtain:

$$\begin{aligned} x_{1,k+1} &\in x_{1,k} + h[x_{2,k} + h(-\alpha x_{2,k} - \partial\Phi(x_{1,k+1}))] \\ &= x_{1,k} + hx_{2,k} - h^2\alpha x_{2,k} - h^2(\frac{x_{1,k+1}}{LC} + \frac{\bar{v}}{LC} - \frac{V_{in}}{2LC}) - h^2\frac{V_{in}}{2LC}\mathrm{Sign}(x_{1,k+1}) \end{aligned} \tag{3.45}$$

or, equivalently,

$$(1 + \frac{h^2}{LC})x_{1,k+1} \in x_{1,k} + hx_{2,k} - h^2\alpha x_{2,k} - h^2(\frac{\bar{v}}{LC} - \frac{V_{in}}{2LC}) - h^2\frac{V_{in}}{2LC}\mathrm{Sign}(x_{1,k+1}).$$

Let

$$a = [x_{1,k} + hx_{2,k} - h^2\alpha x_{2,k} - h^2(\frac{\bar{v}}{LC} - \frac{V_{in}}{2LC})]/(1 + \frac{h^2}{LC})$$

and

$$b = h^2 \frac{V_{in}}{2LC}/(1 + \frac{h^2}{LC}) > 0.$$

Then, we have:

$$x_{1,k+1} \in a - b\mathrm{Sign}(x_{1,k+1}).$$

It is easy to check that:

$$x_{1,k+1} = \begin{cases} a - b & \text{if} \quad a > b, \\ 0 & \text{if} \quad a \in [-b, b], \\ a + b & \text{if} \quad a < -b. \end{cases}$$

We can compute $x_{2,k+1}$ from $x_{1,k+1}$ and (3.44) (Fig. 3.17).

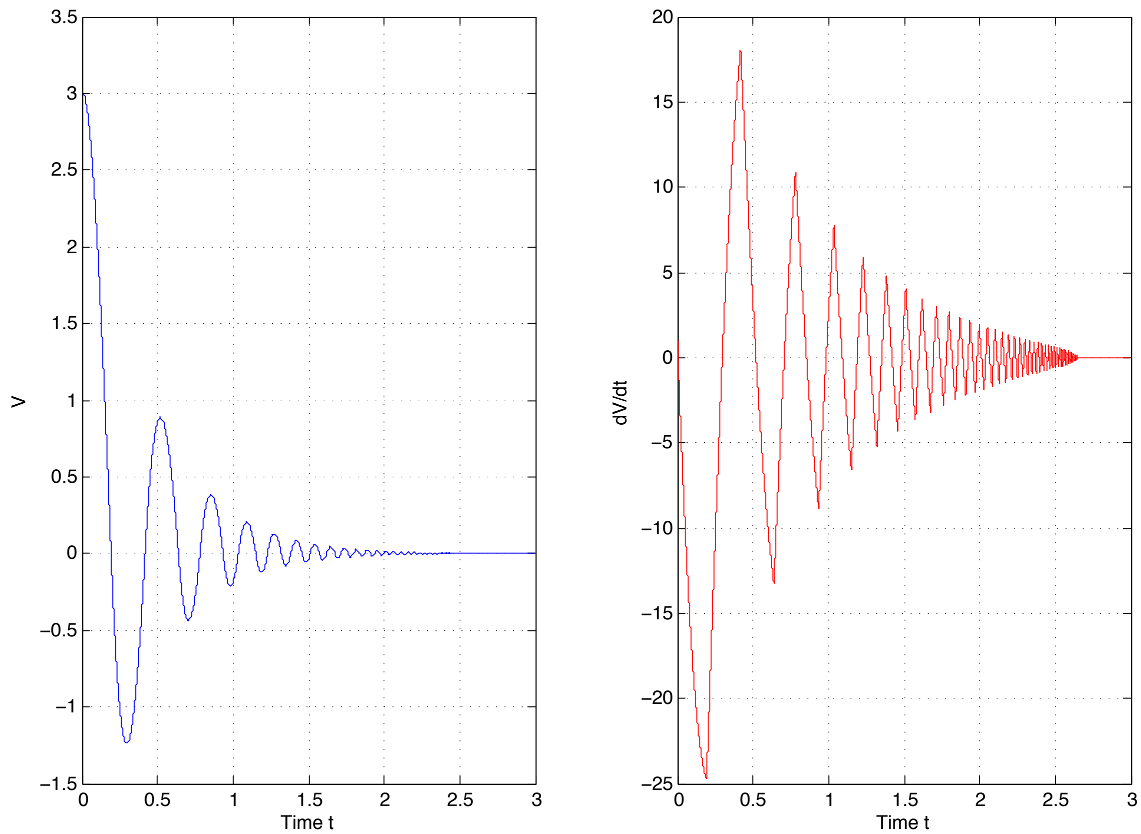


Figure 3.17: Numerical simulation for (3.41) with $V_{in} = 7, V_r = 3$ ($V_r \in [0, V_{in}]$).

3.5 Conclusion

In this chapter, we have presented the ampere-volt characteristics of some electrical devices, including some non- smooth devices like diodes, diacs, silicon controller rectifiers and analyzed some simple RLC circuits with diodes by the tool of complementarity formulation. In the last section, a rigorous mathematical stability analysis of a DC-DC Buck converter is presented by using tools from nonsmooth and variational analysis. The problem is formulated as a nonsmooth dynamical system of Filippov type. The existence of solutions of (3.30) formulated as a first order dynamical system (3.35)-(3.36) is proved. However, the question of uniqueness of solutions remains open. In nonsmooth dynamical systems, the uniqueness of solutions is an important task and has many direct consequences on the behavior of the physical model but it may be not obtained, in general. Some kind of conditions ensuring the uniqueness of solutions are proposed such as hypomonotone, maximal monotone, one-sided Lipschitz, or geometrical conditions from the vector field approach [19, 21, 50, 77]... Unfortunately, these conditions are not satisfied by the system (3.35). Note that the stability analysis and the attractivity results presented in this paper can be also applied to other kind of electronic devices or power converters.

Chapter 4

Lagrange Dynamical Systems with Set-valued Controllers

4.1 Introduction

It is known that the foundation of most of classical mechanics was built by Sir Isaac Newton with his monograph *Philosophiæ Naturalis Principia Mathematica*, published in 1687. His three famous laws of motion have great impact on the scientific vision of physical universe for more than three centuries. In the 1750s, the Swiss mathematician Leonhard Euler and Italian mathematician Joseph Louis Lagrange developed a tool to find functions for which a given functional is stationary by solving a differential equation. This equation is called Euler-Lagrange equation, Euler's equation or Lagrange's equation. The equation is equivalent to the second Newton's law of motion in classical mechanics but it is more convenient for having same form in any system of generalized coordinates and is better for generalization. Indeed, by substituting the expression for the Lagrangian into the Lagrange equation, we obtain the equations of motion of the system. Note that the Lagrangian of the system is not unique even though solving any equivalent Lagrangian yields the same equations of motion [64]. The method is widely used in classical mechanics, economics, electronics, automatics... for solving optimization problems since the functional is stationary at its local extrema, as well as for analyzing desired properties of trajectories of the systems. It plays an important role not only due to broad applications but also for advancing thorough understanding of phenomena in our life.

In order to establish the Lagrange equation for motion of an object, it is necessary to find the *potential energy* $\mathcal{V}(q)$(the stored energy such as when a spring is compressed or an object is lifted up a height) and the *kinetic energy* $\mathcal{T}(q,\dot{q})$ (derived from the motion) with generalized coordinates $q \in \mathbb{R}^n$. Then the Lagrangian $\mathcal{L}(q,\dot{q})$ is defined by the difference between the kinetic energy and the potential energy:

$$\mathcal{L}(q,\dot{q}) = \mathcal{T}(q,\dot{q}) - \mathcal{V}(q),$$

where the kinetic energy is usually given in the quadratic form:

$$\mathcal{T}(q,\dot{q}) = \frac{1}{2}\langle M(q)\dot{q},\dot{q}\rangle,$$

and the potential energy is supposed to be bounded from below. The matrix $M(q) \in \mathbb{R}^{n\times n}$ is called the (mass and/or) inertia matrix, which is positive definite and satisfies some additional conditions that we will verify later. With the generalized coordinates $q \in \mathbb{R}^n$

and the control input $u \in \mathbb{R}^n$, the Lagrange equation has the form:

$$\frac{d}{dt}\big(\frac{\partial \mathcal{L}}{\partial \dot{q}}\big) - \frac{\partial \mathcal{L}}{\partial q} = u. \tag{4.1}$$

Using the first kind of Christoffel symbols [107, 112] to rewrite (4.1) in the form:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla\mathcal{V} = u, \tag{4.2}$$

where $C(q,\dot{q})$ is called the centrifugal and Coriolis (and/or moments) matrix which consists of the terms of centrifugal and Coriolis effects. The term $C(q,\dot{q})\dot{q}$ is the centrifugal and Coriolis force where $\nabla\mathcal{V}$ contains the gravitational forces .We can verify easily that:

$$C(q,\dot{q})\dot{q} = \frac{d}{dt}\big(M(q(t)\big)\dot{q} - \frac{1}{2}\big(\frac{\partial \mathcal{T}(q,\dot{q})}{\partial q}\big). \tag{4.3}$$

The term $\frac{d}{dt}\big(M(q(t)\big) - 2C(q,\dot{q})$ is known then a skew-symmetric matrix [112], where

$$\frac{d}{dt}\big(M(q(t)\big) = \sum_{i=1}^{n} \dot{q}_i \frac{\partial M}{\partial q_i}.$$

Indeed, the (j,k)-entry of the matrix $C(q,\dot{q})$ is given by:

$$C_{jk}(q,\dot{q}) = \sum_{i=1}^{n} C_{ijk}(q)\dot{q}_i, \tag{4.4}$$

where C_{ijk} is defined by the Christoffel symbols of first kind:

$$C_{ijk} = \frac{1}{2}\big(\frac{\partial M_{jk}}{\partial q_i} + \frac{\partial M_{ji}}{\partial q_k} - \frac{\partial M_{ik}}{\partial q_j}\big). \tag{4.5}$$

From (4.4), we can write the matrix $C(q,\dot{q})$ in the form:

$$C(q,\dot{q}) = \sum_{i=1}^{n} \dot{q}_i C_i(q), \tag{4.6}$$

where $C_i(q)$ has the entries $C_{ijk}(q)'s$ and satisfies the equality $C_i + C_i^T = \frac{\partial M}{\partial q_i}$, $i = 1, 2, \ldots, n$. Therefore, we can compute the centrifugal and Coriolis matrix through the inertia matrix and obtain the relationship:

$$\frac{d}{dt}\big(M(q(t)\big) = C(q,\dot{q}) + C(q,\dot{q})^T. \tag{4.7}$$

It is worthy to mention that the equations of many physical systems can be written in the form of Lagrange equation. Various state feedbacks of such systems have been introduced in the literature. In particular, the systems are globally feedback linearizable when they are fully-actuated [83]. Note that measuring all the variables is a necessary condition to perform feedback linearization. Unfortunately in practice, the use of measurement of joint velocities which may be contaminated by noise is not desirable in general. The main obstacle to stabilize and built observers for (4.2) comes from the centrifugal and Coriolis forces. These terms have a quadratic form of velocities and are not measured. Non-affinity property with respect to the unmeasured variables is known to raise many difficulties not only in practice but also in theory [85]. Therefore, it is reasonable to transform the systems to the new ones which are affine in the unmeasured part of the state. G. Besançon in [30] presented a global nonlinear change of coordinates and an observer which converges exponentially for one-degree-of-freedom systems. In [30, 80, 83], the authors have addressed the question that under which conditions, the Lagrange systems can be transformed to be affine in the unmeasured part. That is, the existence of nonsingular matrix T such that:

$$\frac{dT(q)}{dt}\dot{q} = T(q)M^{-1}(q)C(q,\dot{q})\dot{q}. \tag{4.8}$$

In this chapter, we study the systems with a new direction: we are interested in the system (4.2) which is subject to a perturbation force $F(\cdot,q,\dot{q})$ and the control term u is the force of the form $u \in \partial\Phi(\dot{q})$ (for example, the friction force of Coulomb type) where Φ is a scalar convex function to stabilize the system. Therefore, let us consider a class of nonlinear Lagrange dynamical systems with a multivalued controller of the form:

$$M(q(t))\ddot{q}(t) + C(q(t),\dot{q}(t))\dot{q}(t) + \nabla\mathcal{V}(q(t)) + F(t,q(t),\dot{q}(t)) \in -\partial\Phi(\dot{q}(t)) \tag{4.9}$$

for *a.e.* $t \geq t_0$, where $t_0 \in \mathbb{R}$ is fixed, $\Phi : \mathrm{dom}(\Phi) = \mathbb{R}^n \to \mathbb{R}$ is a convex function, $\mathcal{V} \in \mathcal{C}^1(\mathbb{R}^n;\mathbb{R})$ with its gradient $\nabla\mathcal{V}(\cdot)$, the matrices $M(q), C(q,\dot{q}) \in \mathbb{R}^{n\times n}$, and $\partial\Phi(\cdot)$ stands for the convex subdifferential of $\Phi(\cdot)$. Motivated by control applications, we consider in this work the case where the set-valued function in (4.9) depends on the velocity uniquely. Other works [98, 108] have focused on the case when the set-valued part is the normal cone to a subset of $\mathbb{R}^n$, that depends on the position uniquely. This yields different types of dynamics with unilateral constraints, and discontinuities in the velocity. The function perturbation force $F(t,q(t),\dot{q}(t))$ is usually bounded by a constant. The control input $\partial\Phi(\dot{q})$ and gravitational force $\nabla\mathcal{V}$ are applied to stabilize the system (in finite time) at some fixed point. The advantage of such controllers is that they are intrinsically robust since the *a priori* knowledge of the system's parameters (like the inertial parameters) is not necessary for stabilization, and an upperbound of the disturbance is sufficient to reject it. In addition the closed-loop system trajectories attain the equilibrium point in finite-time. Robust discontinuous controller is an important field of research in systems and control [16, 17, 25, 58, 74, 95, 115, 116, 120], where the so-called sliding mode inputs are analyzed. Robust controllers guaranteeing finite-time stability are applied, among many other applications, to mechanical and electro-mechanical systems [54, 67, 71, 110, 117, 121]. Finite-time convergence properties are also studied in the mathematical and control literature [8, 40, 55, 88, 89], as well as stability and invariance properties of nonsmooth systems and differential inclusions [19, 23, 35, 36, 63, 77, 86, 95, 105]. Many of these works are based on the so-called Filippov's differential inclusions and solutions [55]. In other works, maximal monotonicity is the central property, as in [3, 10, 32, 33, 35, 36]. In this chapter the closed-loop dynamics in (4.9) is analyzed. The existence and uniqueness of solutions is first carefully studied, and in a second stage stability properties are examined. The tools are those of convex analysis, which are combined with the dynamical properties of Lagrangian systems. Compared to some previous works dealing with similar dynamics

[10], in this chapter we consider a non-constant mass matrix $M(q)$ in (4.9). As we shall see, this is not a trivial matter because it may destroy the monotonicity of the operator which appears in the first order formulation of (4.9), *i.e.* $z = (z_1, z_2) \mapsto M^{-1}(z_1)\partial\Phi(z_2)$. In other words, the change of variables used in [3, 10, 33, 35, 36] to recast some set-valued systems into maximal monotone differential inclusions, and which uses the chain rule of convex analysis [114, Theorem 4.2.1] no longer works for non-trivial mass matrices. Since uniqueness may not hold for such systems, we propose a stability analysis for also non-unique solutions. Note that, if the matrix $M(q)$ is a constant matrix then from (4.5) and (4.6), we can imply that the centrifugal and Coriolis matrix $C(q, \dot{q}) = 0$. Therefore, our system (4.9) may be considered as a generalization of the system studied in [8] with the following differential inclusion:

$$\ddot{x}(t) + \partial\Phi(\dot{x}(t)) + \nabla f(x(t)) \ni 0, \quad a.e.\ t \geq 0. \tag{4.10}$$

The analysis in this chapter may also be seen as a first step for the study of the digital implementation of such discontinuous controllers, using implicit Euler of zero-order-hold discretizations along the lines of [3, 4]. Such digital implementations allow to suppress the so-called numerical chattering [59, 60], which is a highly undesirable effect in practice, especially in mechanical structures where one wants to decrease as much as possible the vibrations. They also permit to keep in discrete-time the finite-time convergence property, *i.e.* the attractive surfaces are attained after a finite number of steps.

The chapter is divided by 7 sections and organized as follows. Section 4.2 is dedicated to the properties of Lagrange dynamics and characterizes the disturbance as well as an important property of a family of functions $\Phi(\cdot)$. In section 4.3 the existence of solutions issue is tackled through the study of Moreau-Yosida approximations of $\Phi(\cdot)$. The conditions under which uniqueness of solutions hold are examined in section 4.4. Sections 4.5 and 4.6 provide Lyapunov stability and finite-time convergence of the trajectories, respectively. We propose a stability framework that allows for non-unique solutions, as well as an extension of the Krasovskii-LaSalle invariance principle. The sets of switching instants where the velocity attains the zero value are also studied, hence refining the characterization of the solutions. Furthermore, an estimation of the settling-time is provided. Conclusions end the chapter in Section 4.7 finally. This chapter is a joint work with Prof. S. Adly and Prof. B. Brogliato [9].

4.2 Properties of the Lagrangian Dynamics

In this section, we introduce several fundamental assumptions for (4.9) which some of them are important properties of Lagrange systems. They are about the inertia matrix, the Centrifugal and Coriolis matrix, the potential energy, the perturbation force and the control force which usually hold in practice.

Assumption 4.2.1 (The inertia matrix) *:*

($\mathcal{M}_1$) *$M(q)$ is symmetric for all $q \in \mathbb{R}^n$ and there exist $k_1 > 0, k_2 > 0$ such that:*

$$k_1 I_n \leq M(q) \leq k_2 I_n,$$

i.e. for all $q, x \in \mathbb{R}^n$, *we have:* $k_1\|x\|^2 \leq \langle M(q)x, x\rangle \leq k_2\|x\|^2$.

($\mathcal{M}_2$) *There exists* $k_3 > 0$ *such that for all* $u, v, w \in \mathbb{R}^n$*:*

$$\|M(u)w - M(v)w\| \leq k_3\|u - v\|\|w\|.$$

The inertia matrix appears in the calculation of the kinetic energy $\mathcal{T}$ (and also of angular momentum, resultant torque) of the systems by means of the equation:

$$\mathcal{T}(q, \dot{q}) = \frac{1}{2}\langle M(q)\dot{q}, \dot{q}\rangle.$$

From ($\mathcal{M}_1$), we have that $M(q)$ is positive definite and $\frac{k_1}{2}\|\dot{q}\|^2 \leq \mathcal{T}(q, \dot{q}) \leq \frac{k_2}{2}\|\dot{q}\|^2$ for all $q, \dot{q} \in \mathbb{R}^n$. The assumption ($\mathcal{M}_2$) may be interpreted as the Lipschitz property of the inertia matrix. It is noteworthy that $k_1 > 0$ may not always be satisfied since it is well-known that some generalized coordinate changes may yield singularities in the mass matrix. For large classes of systems like kinematic chains (manipulators) this assumption holds, however.

Assumption 4.2.2 (The centrifugal and Coriolis matrix) *:*

($\mathcal{C}_1$) *There exists* $k_4 > 0$ *such that for all* $u, v \in \mathbb{R}^n$ *:*

$$\|C(u, v)\|_m \leq k_4\|v\|.$$

($\mathcal{C}_2$) *For all* $t \geq t_0$, *we have:*

$$\frac{d}{dt}(M(q(t))) = C(q(t), \dot{q}(t)) + C(q(t), \dot{q}(t))^T.$$

The centrifugal and Coriolis matrix contains the terms of centrifugal and Coriolis effects, the forces and/or moments. In particular Assumption ($\mathcal{C}_2$) holds if $C(q, \dot{q})$ is defined from the so-called Christoffel's symbols of the first kind as we have seen in (4.7). It means that $\frac{d}{dt}(M(q(t))) - 2C(q(t), \dot{q}(t))$ is anti-symmetric and hence for all $u \in \mathbb{R}^n$:

$$\left\langle u, \{\frac{d}{dt}(M(q(t))) - 2C(q(t), \dot{q}(t))\}u\right\rangle = 0. \tag{4.11}$$

Assumption 4.2.3 *:*

($\mathcal{H}_\Phi$) $\min_{x\in\mathbb{R}^n} \Phi(x) = \Phi(0) = 0$.

($\mathcal{H}_C$) *The function* $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *defined by* $h(x_1, x_2) = C(x_1, x_2)x_2$ *is locally Lipschitz.*

($\mathcal{H}_F$) *The function* $F(t, x_1, x_2)$ *from* $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous in* t, *uniformly locally Lipschitz in* x_1, x_2 *(i.e. the Lipschitz constant is independent of* t*) and bounded.*

$(\mathcal{H}_{\Phi,F})$ *There exists* $\lambda^* > 0$ *such that for all* $t \geq 0, x, y \in \mathbb{R}^n$ *and* $0 < \lambda \leq \lambda^*$ *we have:*

$$\langle \nabla \Phi_\lambda(y) + F(t,x,y), y \rangle \geq 0, \tag{4.12}$$

where $\Phi_\lambda(\cdot)$ *is the Moreau approximation of* $\Phi(\cdot)$ *for each* $\lambda > 0$, *i.e.,*

$$\Phi_\lambda(\cdot) := \inf_{z \in R^n} \{\Phi(z) + \frac{1}{2\lambda} \|z - \cdot\|^2\}.$$

$(\mathcal{H}_\mathcal{V} - i)$ $\mathcal{V}(\cdot)$ *is* $\mathcal{C}^1$ *and* $\nabla \mathcal{V}(\cdot)$ *is Lipschitz continuous on bounded sets.*

$(\mathcal{H}_\mathcal{V} - ii)$ $\mathcal{V}(\cdot)$ *is bounded from below.*

Let us recall the Moreau-Yosida approximation Theorem for a maximal monotone operator. Note that $\partial\Phi$ is maximal monotone since Φ is a convex function on $\mathbb{R}^n$. Then $\Phi_\lambda, \nabla\Phi_\lambda$ are the approximations of $\Phi, \partial\Phi$ respectively that $\Phi_\lambda \to \Phi$ (pointwisely) and $\nabla\Phi_\lambda$ converges to $m(\partial\Phi)$, the minimal norm function of $\partial\Phi$. Indeed, $\nabla\Phi_\lambda$ is the Yosida approximation of the maximal monotone operator $\partial\Phi$. Furthermore, we have $\partial\Phi$ is locally bounded and its graph (in finite dimension $\mathbb{R}^n$) is closed. It implies that $\partial\Phi$ is also upper semicontinuous [19].

Theorem 4.2.1 *(Moreau-Yosida approximation [19]) Let* $\mathcal{F}(\cdot)$ *be maximal monotone and let* $\lambda > 0$. *Then:*
1) *the* resolvent *of* $\mathcal{F}(\cdot)$ *(of index* λ*) defined by* $J_\lambda = (I + \lambda\mathcal{F})^{-1}$ *is a non-expansive single valued map from* X *to* X.
2) *the* Moreau-Yosida approximation *(of index* λ*) of* $\mathcal{F}(\cdot)$ *defined by* $\mathcal{F}_\lambda = \frac{1}{\lambda}(I - J_\lambda)$ *satisfies:*
   i) for all $x \in X$, $\mathcal{F}_\lambda(x) \in \mathcal{F}(J_\lambda x)$,
   ii) $\mathcal{F}_\lambda(\cdot)$ *is Lipschitz with constant* $\frac{1}{\lambda}$ *and maximal monotone.*
3) *For all* $x \in \mathrm{Dom}(\mathcal{F})$*:*
   i) $J_\lambda(x)$ *converges to* x,
   ii) $\mathcal{F}_\lambda(x)$ *converges to* $m(\mathcal{F}(x))$, *where* $m(\mathcal{F}(x))$ *is the element of* $\mathcal{F}(x)$ *of minimal norm.*

The Moreau-Yosida approximation $\mathcal{F}_\lambda$ of $\mathcal{F}$ is an approximation selection in the following sense:

$$\forall x \in \mathrm{Dom}(\mathcal{F}),\ (x, \mathcal{F}_\lambda(x)) \in \mathrm{Graph}(\mathcal{F}) + m(\mathcal{F}(x))\mathbb{B}_\lambda. \tag{4.13}$$

Remark 4.2.1 *Assumptions* $(\mathcal{M}_1)$, $(\mathcal{M}_2)$, $(\mathcal{C}_1)$, $(\mathcal{C}_2)$, $(\mathcal{H}_C)$ *are used for the design of stabilizing controllers in Robotics [18, 112]. They are typically used, together with* $(\mathcal{H}_\mathcal{V} - ii)$, *to prove the dissipativity of the Lagrange dynamics [38]. The potential energy* $\mathcal{V}$ *is smooth and bounded from below in general. The function* $F(\cdot)$ *plays a role of perturbation force which is usually bounded by a constant. If the system is not subject to disturbances, i.e.* $F(\cdot) \equiv 0$, *the property* $(\mathcal{H}_{\Phi,F})$ *of Assumption 4.2.3 naturally holds for all* $\lambda^* > 0$. *It means that if module of perturbation force is small enough, this property holds for a large class of function* Φ.

The following lemma shows how to compute the Moreau–Yosida approximation of the Euclidean and 1-norm functions. This result will be used in Lemma 4.2.2 to gives some cases where the property $(\mathcal{H}_{\Phi,F})$ is satisfied.

Lemma 4.2.1 *The following holds: (i) if* $\Phi(\cdot) = \|\cdot\|$ *then* $\Phi_\lambda(y) = \frac{\|y\|^2}{2\lambda}$ *for* $\|y\| \le \lambda$ *and* $\Phi_\lambda(y) = \|y\| - \frac{\lambda}{2}$ *for* $\|y\| > \lambda$.
(ii) if $\Phi(\cdot) = \|\cdot\|_1$ *then* $\Phi_\lambda(y) = \frac{\|y\|_1^2}{2\lambda}$ *for* $\|y\|_1 \le \lambda$ *and* $\Phi_\lambda(y) = \|y\|_1 - \frac{\lambda}{2}$ *for* $\|y\|_1 > \lambda$.
(iii) if $\Phi(\cdot) = \max\{\|\cdot\|, \|\cdot\|^2\}$ *then:*

$$\Phi_\lambda(y) = \begin{cases} \frac{\|y\|^2}{2\lambda} & \text{if} \quad \|y\| \le \lambda, \\ \|y\| - \frac{\lambda}{2} & \text{if} \quad \lambda \le \|y\| \le 1+\lambda, \\ \frac{(\|y\|-1)^2}{2\lambda} + 1 & \text{if} \quad 1+\lambda \le \|y\| \le 1+2\lambda, \\ \frac{\|y\|^2}{2\lambda+1} & \text{if} \quad \|y\| \ge 1+2\lambda. \end{cases}$$

Proof. (i) For each $y \in \mathbb{R}^n$, let $R_y(z) := \|z\| + \frac{1}{2\lambda}\|z-y\|^2 = \|z\| + \frac{1}{2\lambda}(\|z\|^2 - 2\langle z, y\rangle + \|y\|^2),\ z \in \mathbb{R}^n$. For $\|y\| \le \lambda$, we have $\|\frac{1}{\lambda}\langle z, y\rangle\| \le \|z\|$. Hence, $R_y(z) \ge \frac{\|y\|^2}{2\lambda}$ and $\Phi_\lambda(y) = R_y(0) = \frac{\|y\|^2}{2\lambda}$. For $\|y\| > \lambda$, it is easy to see that $R_y(z) \ge \frac{1}{2\lambda}\{\|z\|^2 - 2(\|y\|-\lambda)\|z\| + \|y\|^2\} = \frac{1}{2\lambda}(\|z\| - \|y\| + \lambda)^2 + \|y\| - \frac{\lambda}{2} \ge \|y\| - \frac{\lambda}{2}$. Therefore $\Phi_\lambda(y) = R_y(z') = \|y\| - \frac{\lambda}{2}$ where $z' \in \mathbb{R}^n$ satisfies $\langle z', y\rangle = \|z'\|\|y\|$ and $\|z'\| + \lambda = \|y\|$.

ii) Let $R_y^1(z) := \|z\|_1 + \frac{1}{2\lambda}\|z-y\|_1^2 = |z_1| + \ldots + |z_n| + \frac{1}{2\lambda}(|z_1 - y_1| + \ldots + |z_n - y_n|)^2$. Since we want to minimize $R_y^1(z)$, we may consider $z \in \mathbb{R}^n$ which satisfies $y_i z_i \ge 0,\ |z_i| \le |y_i|$ for $i = 1, \ldots, n$ (*). Then $R_y^1(z) \ge |z_1| + \ldots + |z_n| + \frac{1}{2\lambda}(|y_1| - |z_1| + \ldots + |y_n| - |z_n|)^2 = \|z\|_1 + \frac{1}{2\lambda}(\|y\|_1 - \|z\|_1)^2$. If $\|y\|_1 \le \lambda$ then $R_y^1(z) \ge \|z\|_1 - \frac{\|y\|_1}{\lambda}\|z\|_1 + \frac{1}{2\lambda}(\|y\|_1^2 + \|z\|_1^2) \ge \frac{\|y\|_1^2}{2\lambda}$. Therefore, $\Phi_\lambda(y) = R_y^1(0) = \frac{\|y\|_1^2}{2\lambda}$. If $\|y\|_1 > \lambda$, then $R_y^1(z) \ge \|z\|_1 + \frac{1}{2\lambda}(\|y\|_1 - \|z\|_1)^2 = \frac{1}{2\lambda}(\|z\|_1 - \|y\|_1 + \lambda)^2 + \|y\|_1 - \frac{\lambda}{2} \ge \|y\|_1 - \frac{\lambda}{2}$. Hence, $\Phi_\lambda(y) = R_y^1(z') = \|y\|_1 - \frac{\lambda}{2}$ where z' satisfies condition $(*)$ and $\|z'\|_1 + \lambda = \|y\|_1$.

iii) For each $y \in \mathbb{R}^n$, let $R_y^*(z) := \Phi(z) + \frac{1}{2\lambda}\|z-y\|^2, z \in \mathbb{R}^n$. For $\|y\| \le \lambda$, we have $R_y^*(z) \ge R_y(z) \ge R_y(0) = R_y^*(0)$. Hence $\Phi_\lambda(y) = R_y(0) = \frac{\|y\|^2}{2\lambda}$. If $\lambda \le \|y\| \le 1+\lambda$, then $R_y^*(z) \ge R_y(z) \ge R_y(z') = R_y^*(z') = \|y\| - \frac{\lambda}{2}$ where $z' \in \mathbb{R}^n$ satisfies $\langle z', y\rangle = \|z'\|\|y\|$ and $\|z'\| + \lambda = \|y\|$ (see the arguments in (i)). Therefore, $\Phi_\lambda(y) = \|y\| - \frac{\lambda}{2}$. If $\|y\| \ge 1+2\lambda$, it is easy to see that $R_y^*(z) \ge z^2 + \frac{1}{2\lambda}\|z-y\|^2 = \frac{2\lambda+1}{2\lambda}(z - \frac{y}{2\lambda+1})^2 + \frac{\|y\|^2}{2\lambda+1} \ge \frac{\|y\|^2}{2\lambda+1} = R_y^*(z_1)$, where $z_1 = \frac{y}{2\lambda+1}$. So in this case, $\Phi_\lambda(y) = \frac{\|y\|^2}{2\lambda+1}$. In the remain case, if $1+\lambda \le \|y\| \le 1+2\lambda$, it is not difficult to verify that $R_y^*(z)$ attains its minimum on the unit sphere. Hence, $\Phi_\lambda(y) = 1 + \frac{1}{2\lambda}(1 - 2\|y\| + \|y\|^2) = 1 + \frac{(\|y\|-1)^2}{2\lambda} = R_y^*(z_2)$, where z_2 belongs to the unit sphere and has the same direction with y. ■

Lemma 4.2.2 *Let i)* $\Phi(\cdot) = c\|\cdot\|$, *ii)* $\Phi(\cdot) = c\|\cdot\|_1$ *or iii)* $\Phi(\cdot) = c\max\{\|\cdot\|, \|\cdot\|^2\}$, *where* $c > 0$. *Then if*

$$\sup_{(t,x_1)\in\mathbb{R}\times\mathbb{R}^n} \|F(t, x_1, \cdot)\| \leq c\min\{1, \frac{\|\cdot\|}{\lambda'}\}$$

for some $\lambda' > 0$, *property* $(\mathcal{H}_{\Phi,F})$ *is satisfied.*

Proof. We choose $\lambda^* := \lambda'$.

i) First, let us consider $\Phi(\cdot) = c\|\cdot\|$. Then for $0 < \lambda \leq \lambda^*$, $t \geq 0, x \in \mathbb{R}^n$, $\|y\| > \lambda$, by using Lemma 4.2.1, we have $\langle\nabla\Phi_\lambda(y)+F(t,x,y), y\rangle = \langle\frac{cy}{\|y\|}+F(t,x,y), y\rangle \geq c\|y\|-c\|y\| = 0$. In the case $\|y\| \leq \lambda$, we also have $\langle\nabla\Phi_\lambda(y)+F(t,x,y), y\rangle = \langle\frac{cy}{\lambda}+F(t,x,y), y\rangle \geq \frac{c\|y\|^2}{\lambda^*} - \frac{c\|y\|^2}{\lambda^*} = 0$.

ii) If $\Phi(\cdot) = c\|\cdot\|_1$, the computations can be done similarly. Indeed, for $0 < \lambda \leq \lambda^*$, $t \geq 0, x \in \mathbb{R}^n, \|y\|_1 > \lambda$, we have $\langle\nabla\Phi_\lambda(y) + F(t,x,y), y\rangle = c\|y\|_1 + \langle F(t,x,y), y\rangle \geq c\|y\|_1 - c\|y\| \geq 0$. When $\|y\|_1 \leq \lambda$, we obtain $\langle\nabla\Phi_\lambda(y) + F(t,x,y), y\rangle = \frac{c\|y\|_1^2}{\lambda} + \langle F(t,x,y), y\rangle \geq \frac{c\|y\|_1^2}{\lambda^*} - \frac{c\|y\|^2}{\lambda^*} \geq 0$.

iii) If $\Phi(\cdot) = c\max\{\|\cdot\|, \|\cdot\|^2\}$, the case of $\|y\| \leq 1+\lambda$ can be proved similarly to *i*). If $1+\lambda \leq \|y\| \leq 1+2\lambda$, we obtain: $\langle\nabla\Phi_\lambda(y) + F(t,x,y), y\rangle = \langle\frac{cy}{\lambda} - \frac{cy}{\|y\|\lambda} + F(t,x,y), y\rangle \geq c(\frac{\|y\|^2}{\lambda} - \frac{\|y\|}{\lambda} - \|y\|) \geq c(\frac{\|y\|(1+\lambda)}{\lambda} - \frac{\|y\|}{\lambda} - \|y\|) = 0$. If $\|y\| \geq 1+2\lambda$ then $\langle\nabla\Phi_\lambda(y) + F(t,x,y), y\rangle = \langle\frac{2cy}{2\lambda+1} + F(t,x,y), y\rangle \geq \frac{2c\|y\|^2}{2\lambda+1} - c\|y\| \geq 2c\|y\| - c\|y\| \geq 0$, since $\|y\| \geq 1+2\lambda$. ■

In the scalar case, the Moreau-Yosida approximation is depicted in Figure 4.1. One recovers the saturation function that is widely used in control applications. Considering the Moreau-Yosida approximation associated with the Euclidean norm and 1-norm are interesting because it allows us to study in a systematic way the extension of the saturation function towards codimension ≥ 2 switching surfaces.

The matrix $M^{-1}(q)$ exists and is symmetric positive definite for all q since $M(q)$ is symmetric positive definite for all q. Furthermore, we have:

Lemma 4.2.3 *For all* $q \in \mathbb{R}^n$:

$$\frac{1}{k_2} \leq \|M^{-1}(q)\|_m \leq \frac{1}{k_1},$$

where the norm used here is the induced matrix norm.

Proof. For all $x \in \mathbb{R}^n$, we have:

$$\begin{aligned}\langle M^{-1}(q)x, x\rangle &= x^TM^{-1}(q)x = x^TM^{-1}(q)M(q)M^{-1}(q)x \\ &= (M^{-1}(q)x)^TM(q)(M^{-1}(q)x) \geq k_1\|M^{-1}(q)x\|^2\end{aligned}$$

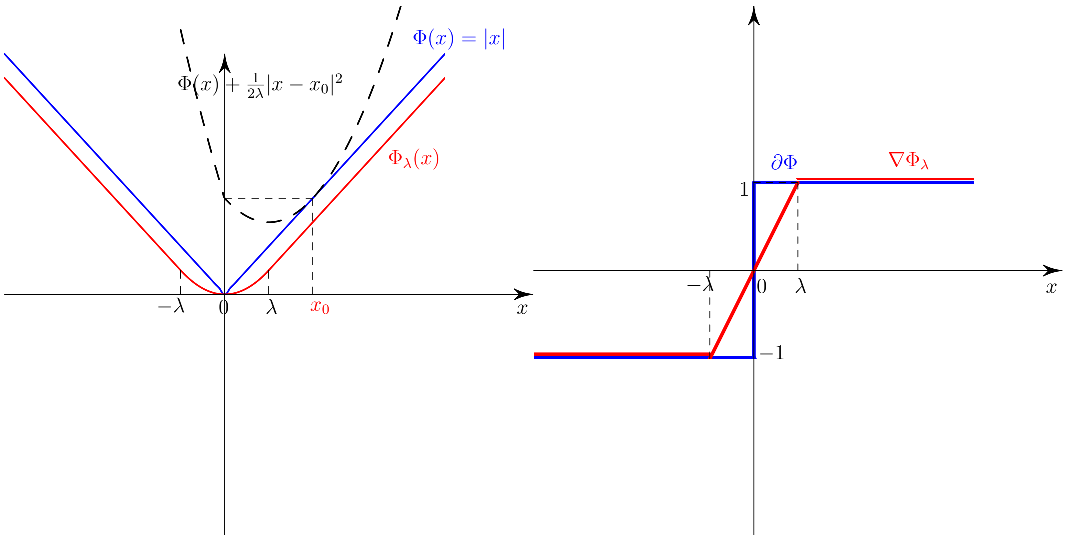


Figure 4.1: Moreau-Yosida approximation of $\Phi(x) = |x|,\ x \in \mathbb{R}$.

$\Rightarrow k_1 \|M^{-1}(q)x\|^2 \le \langle M^{-1}(q)x, x\rangle \le \|M^{-1}(q)x\| \|x\|$

$\Rightarrow \frac{\|M^{-1}(q)x\|}{\|x\|} \le \frac{1}{k_1} \Rightarrow \|M^{-1}(q)\|_m \le \frac{1}{k_1}$.

Let x_0 be an eigenvector of $M^{-1}(q)$ and $\|x_0\| = 1$, we obtain:

$$\langle M^{-1}(q)x_0, x_0\rangle = \|M^{-1}(q)x_0\|.$$

Then

$$\|M^{-1}(q)x_0\| = \langle M^{-1}(q)x_0, x_0\rangle = (M^{-1}(q)x_0)^T M(q)(M^{-1}(q)x_0) \le k_2 \|M^{-1}(q)x_0\|^2$$
$$\Rightarrow \|M^{-1}(q)x_0\| \ge \frac{1}{k_2} \Rightarrow \|M^{-1}(q)\|_m = \sup_{\|x\|=1} \|M^{-1}(q)x\| \ge \frac{1}{k_2}.$$

We have the result. ■

Lemma 4.2.4 *There exists $k_5 > 0$ such that for all $u, v, w \in \mathbb{R}^n$:*

$$\|M^{-1}(u)w - M^{-1}(v)w\| \le k_5 \|u - v\| \|w\|.$$

Proof. We have:

$$\|M^{-1}(u)w - M^{-1}(v)w\| = \|M^{-1}(u)w - M^{-1}(u)M(u)M^{-1}(v)w\|$$
$$\le \|M^{-1}(u)\|_m . \|w - M(u)M^{-1}(v)w\| \le \frac{1}{k_1}\|M(v)M^{-1}(v)w - M(u)M^{-1}(v)w\|$$
$$\le \frac{k_3}{k_1}\|u - v\|.\|M^{-1}(v)w\| \le \frac{k_3}{k_1^2}\|u - v\|.\|w\|.$$

Hence we obtain the result with $k_5 = \frac{k_3}{k_1^2}$. ■

4.3 Existence of Solutions

For any positive real number λ, we approximate the differential inclusion (4.9) by the following differential equation:

$$M(q_\lambda(t))\ddot{q_\lambda}(t)+C(q_\lambda(t),\dot{q_\lambda}(t))\dot{q_\lambda}(t)+\nabla\mathcal{V}(q_\lambda(t))+F(t,q_\lambda(t),\dot{q_\lambda}(t)) = -\nabla\Phi_\lambda(\dot{q_\lambda}(t)), \quad (4.14)$$

where $\Phi_\lambda(\cdot)$ denotes the Moreau approximation (of index λ) of $\Phi(\cdot)$. For all $\lambda > 0$, we have $\Phi_\lambda(\cdot)$ is $\mathcal{C}^1$ and $\nabla\Phi_\lambda(\cdot)$ is Lipschitz continuous with constant $\frac{1}{\lambda}$. Without loss of generality, we suppose from now that $t_0 = 0$. Then, for given initial condition, by using Cauchy-Lipschitz Theorem and dissipation property of the systems, we can obtain the existence and uniqueness of classical solution of the approximation differential equation (4.14) for each λ. There is a subsequence of such classical solutions which converges to the solution of (4.9) in the sense of Theorem 4.3.1. We begin with the following lemma for the approximation equation and then with the existence theorem for our original system.

Lemma 4.3.1 *Let Assumptions* 4.2.1, 4.2.2, 4.2.3 *hold and* $0 < \lambda \le \lambda^*$ *(*λ^* *is defined in Assumption* 4.2.3*). Then, for every* $(q_0, \dot{q}_0) \in \mathbb{R}^n \times \mathbb{R}^n$*, there exists a unique maximal (classical) solution* $q_\lambda : [0, +\infty) \to \mathbb{R}^n$ *satisfying* $(q_\lambda(0), \dot{q_\lambda}(0)) = (q_0, \dot{q}_0)$.

Proof. Let $x_1 = q_\lambda, x_2 = \dot{q_\lambda}$ and $x = (x_1^T\ x_2^T)^T$. We rewrite (3.42) to the first-order ODE:

$$\dot{x} = f(t, x), \quad (4.15)$$

where $f : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and

$$f(t,x) = \begin{pmatrix} x_2 \\ -M^{-1}(x_1)[\nabla\Phi_\lambda(x_2) + \nabla\mathcal{V}(x_1) + C(x_1,x_2)x_2 + F(t,x_1,x_2)] \end{pmatrix}. \quad (4.16)$$

Let $g : \mathbb{R}^{2n} \to \mathbb{R}^n$ and

$$g(x) = g(x_1, x_2) := -[\nabla\Phi_\lambda(x_2) + \nabla\mathcal{V}(x_1) + C(x_1,x_2)x_2], \quad (4.17)$$

then $g(\cdot)$ is locally Lipschitz due to the assumptions $(\mathcal{H}_C), (\mathcal{H}_\mathcal{V} - i)$. Let

$$G(x) = G(x_1, x_2) := M^{-1}(x_1)g(x). \quad (4.18)$$

We will prove that $G(\cdot)$ is also locally Lipschitz. In fact, for all $x, y \in \mathbb{R}^{2n}$, we have :

$$\begin{aligned}&\|G(x) - G(y)\| = \|M^{-1}(x_1)g(x) - M^{-1}(y_1)g(y)\| \le \|M^{-1}(x_1)g(x) - M^{-1}(y_1)g(x)\| \\ &+\|M^{-1}(y_1)g(x) - M^{-1}(y_1)g(y)\| \le k_5\|x_1 - y_1\|\|g(x)\| + \|M^{-1}(y_1)\|_m\|g(x) - g(y)\| \\ &\le k_5\|x_1 - y_1\|\|g(x)\| + \frac{1}{k_1}\|g(x) - g(y)\|.\end{aligned}$$

Therefore, $G(\cdot)$ is locally Lipschitz. Combining with the assumption $(\mathcal{H}_F)$, we obtain that $f(t,x)$ is continuous in t and locally Lipschitz in x. Then for given $(q_{\lambda 0}, \dot{q}_{\lambda 0})$, there exists uniquely a $\mathcal{C}^1$ local solution for (4.14), or equivalently (4.15) (Cauchy-Lipschitz Theorem). Let $(q_\lambda(t), \dot{q_\lambda}(t))$ be the maximal solution defined on some interval $[0, T_{\max})$ with $0 < T_{\max} \le$

$+\infty$. Consider the energy function, which is the sum of the kinetic energy and the (virtual) potential energy:

$$V(q_\lambda, \dot{q_\lambda}) := \frac{1}{2}\langle M(q_\lambda)\dot{q_\lambda}, \dot{q_\lambda}\rangle + \mathcal{V}(q_\lambda). \tag{4.19}$$

Then the derivative of $V(\cdot)$ along the trajectories of (4.14) is:

$$\begin{aligned}
\dot{V}(q_\lambda(t), \dot{q_\lambda}(t)) &= \frac{1}{2}\langle \frac{d}{dt}(M(q_\lambda(t)))\dot{q_\lambda}(t), \dot{q_\lambda}(t)\rangle + \langle M(q_\lambda(t))\dot{q_\lambda}(t), \ddot{q_\lambda}(t)\rangle + \langle \nabla\mathcal{V}(q_\lambda(t)), \dot{q_\lambda}(t)\rangle \\
&= \frac{1}{2}\langle \frac{d}{dt}(M(q_\lambda(t)))\dot{q_\lambda}(t), \dot{q_\lambda}(t)\rangle - \langle C(q_\lambda, \dot{q_\lambda})\dot{q_\lambda} + \nabla\mathcal{V}(q_\lambda) + \nabla\Phi_\lambda(\dot{q_\lambda}) \\
&+ F(t, q_\lambda(t), \dot{q_\lambda}(t)), \dot{q_\lambda}(t)\rangle + \langle \nabla\mathcal{V}(q_\lambda(t)), \dot{q_\lambda}(t)\rangle \\
&= \frac{1}{2}\langle [\frac{d}{dt}(M(q_\lambda(t))) - 2C(q_\lambda, \dot{q_\lambda})]\dot{q_\lambda}(t), \dot{q_\lambda}(t)\rangle \\
&\quad -\langle \nabla\Phi_\lambda(\dot{q_\lambda}(t)) + F(t, q_\lambda(t), \dot{q_\lambda}(t)), \dot{q_\lambda}(t)\rangle \\
&= -\langle \nabla\Phi_\lambda(\dot{q_\lambda}(t)) + F(t, q_\lambda(t), \dot{q_\lambda}(t)), \dot{q_\lambda}(t)\rangle \le 0,
\end{aligned}$$

for almost all $t \ge 0$, where Assumption 4.2.2 ($\mathcal{C}_2$) and Assumption 4.2.3 ($\mathcal{H}_{\Phi,F}$) are used. Therefore for all $t \ge 0$, from Assumption 4.2.1 ($\mathcal{M}_1$) and Assumption 4.2.3 ($\mathcal{H}_\mathcal{V} - ii$), we obtain:

$$\begin{aligned}
\frac{1}{2}k_1\|\dot{q_\lambda}(t)\|^2 &\le \frac{1}{2}\langle M(q_\lambda(t))\dot{q_\lambda}(t), \dot{q_\lambda}(t)\rangle = V(q_\lambda(t), \dot{q_\lambda}(t)) - \mathcal{V}(q_\lambda)(t) \\
&\le V(q_0, \dot{q}_0) - \inf \mathcal{V},
\end{aligned} \tag{4.20}$$

which implies that $\dot{q_\lambda}$ is bounded. Assume that $T_{\max} < +\infty$, we have:

$$\|q_\lambda(t)\| \le \|q_0\| + T_{\max} \sup_{t\in[0,T_{\max})} \|\dot{q_\lambda}\| < +\infty.$$

Hence, $(q_\lambda(t), \dot{q_\lambda}(t))$ is bounded on $[0, T_{\max})$. Our result follows by contradiction. ■

Remark 4.3.1 *We can see that it is the perturbation force F which affects some nice properties of the system. Without F, it is obvious that the function V is decreasing along the trajectories for all $\lambda > 0$ due to the non-positivity of the orbital derivative. The assumption ($\mathcal{H}_{\Phi,F}$) plays a role in keeping the validity of the Lyapunov function. It would be more elegant if in ($\mathcal{H}_{\Phi,F}$) the inequality does not depend on λ. One of the ideas is that we can approximate F by F_λ in the differential equation (4.14) where $F_\lambda \to F$ and F_λ is continuous in t, uniformly locally Lipschitz in x_1, x_2; for example F_λ satisfies the following equality:*

$$\langle F_\lambda(t, x, y), y\rangle = \langle F(t, x, y), y\rangle + \Phi(y) - \Phi_\lambda(y) \tag{4.21}$$

for all $t \ge 0$ and $x, y \in \mathbb{R}^n$. Indeed, we can verify that then we can replace the assumption ($\mathcal{H}_{\Phi,F}$) by the new one:

($\mathcal{H}'_{\Phi,F}$) *for all $t \ge 0$ and $x, y \in \mathbb{R}^n$:*

$$\Phi(y) + \langle F(t, x, y), y\rangle \ge 0. \tag{4.22}$$

by using the fact that:

$$\langle \nabla\Phi_\lambda(y), y\rangle \ge \Phi_\lambda(y) - \Phi_\lambda(0) = \Phi_\lambda(y).$$

Theorem 4.3.1 *Let Assumptions* 4.2.1, 4.2.2, 4.2.3 *hold. Then, for every* $(q_0, \dot{q}_0) \in \mathbb{R}^n \times \mathbb{R}^n$*, there exists a solution* $q : [0, +\infty) \to \mathbb{R}^n$ *of* (4.9) *in the following sense:*
(a) $q \in \mathcal{C}^1([0, +\infty), \mathbb{R}^n) \cap \mathcal{W}^{2,\infty}([0, T], \mathbb{R}^n)$ *for every* $T > 0$.
(b) (4.9) *is satisfied for a.e.* $t \in [0, +\infty)$.
(c) $q(0) = q_0$ *and* $\dot{q}(0) = \dot{q}_0$.

Proof. Consider $\lambda \in [0, \lambda^*]$. We desire the convergence of a subsequence of solutions of (4.14) to a function expected to be the solution of the original system (4.9). Indeed, from (4.20), we have:

$$(\dot{q_\lambda}) \text{ is uniformly bounded in } \mathrm{L}^\infty([0, +\infty), \mathbb{R}^n). \tag{4.23}$$

Given a fix $T > 0$. From $q_\lambda(t) = q_0 + \int_0^t \dot{q_\lambda}(s)ds$ we obtain:

$$(q_\lambda) \text{ is uniformly bounded in } \mathrm{L}^\infty([0, \mathrm{T}], \mathbb{R}^n). \tag{4.24}$$

Since $\nabla\mathcal{V}(\cdot)$ is bounded on bounded sets, this implies:

$$(\nabla\mathcal{V}(q_\lambda)) \text{ is uniformly bounded in } \mathrm{L}^\infty([0, \mathrm{T}], \mathbb{R}^n). \tag{4.25}$$

Furthermore, from Assumption 4.2.2 $(\mathcal{C}_1)$, we have $\|C(q_\lambda(t), \dot{q_\lambda}(t))\dot{q_\lambda}(t)\| \leq k_4\|\dot{q_\lambda}(t)\|^2$ which gives:

$$(C(q_\lambda(t), \dot{q_\lambda})\dot{q_\lambda}) \text{ is uniformly bounded in } \mathrm{L}^\infty([0, \mathrm{T}], \mathbb{R}^n). \tag{4.26}$$

Finally, it is classical that for all $x \in \mathbb{R}^n$, $\|\nabla\Phi_\lambda(x)\| \leq \|m(\partial\Phi(x))\|$ where $m(\partial\Phi(x))$ is the element of minimal norm. Hence, $\|\nabla\Phi_\lambda(\dot{q_\lambda})\| \leq \|m(\partial\Phi(\dot{q_\lambda}))\|$. Note that since $\partial\Phi(\cdot)$ is bounded on bounded sets, we obtain:

$$(\nabla\Phi_\lambda(\dot{q_\lambda})) \text{ is uniformly bounded in } \mathrm{L}^\infty([0, \mathrm{T}], \mathbb{R}^n). \tag{4.27}$$

From (4.9) and Assumption 4.2.3 $(\mathcal{H}_F)$, we have:

$$(\ddot{q_\lambda}) \text{ is uniformly bounded in } \mathrm{L}^\infty([0, \mathrm{T}], \mathbb{R}^n). \tag{4.28}$$

Then there exists a function $q \in \mathcal{C}^1([0, T], \mathbb{R}^n) \cap \mathcal{W}^{2,+\infty}([0, T], \mathbb{R}^n)$ and a subsequence of (q_λ), still denoted by (q_λ) such that $q_\lambda(\cdot), \dot{q_\lambda}(\cdot)$ converge uniformly to $q(\cdot), \dot{q}(\cdot)$ respectively, and $\ddot{q_\lambda}(\cdot)$ converges to $\ddot{q}(\cdot)$ for the topology $\sigma(L^\infty([0, T], \mathbb{R}^n), L^1([0, T], \mathbb{R}^n))$ (see [8]). We prove that $q(\cdot)$ satisfies (4.9) for a.e. t on $[0, T]$. With the same arguments as in [8], it is sufficient to prove that:

$$M(q_\lambda)\ddot{q_\lambda} + C(q_\lambda, \dot{q_\lambda})\dot{q_\lambda} + \nabla\mathcal{V}(q_\lambda) + F(\cdot, q_\lambda, \dot{q_\lambda}) \to M(q)\ddot{q} + C(q, \dot{q})\dot{q} + \nabla\mathcal{V}(q) + F(\cdot, q, \dot{q})$$

for the topology $\sigma(L^\infty, L^1)$.
Since $q_\lambda \to q$ uniformly and $\nabla\mathcal{V}(\cdot)$ is Lipschitz continuous on the bounded sets, we have $\nabla\mathcal{V}(q_\lambda) \to \nabla\mathcal{V}(q)$ uniformly in $\mathcal{C}([0, T], \mathbb{R}^n)$. From Assumption 4.2.3 $(\mathcal{H}_C), (\mathcal{H}_F)$ and the fact that $q_\lambda \to q, \dot{q_\lambda} \to \dot{q}$, we obtain that $C(q_\lambda, \dot{q_\lambda})\dot{q_\lambda} + F(\cdot, q_\lambda, \dot{q_\lambda})$ converges to $C(q, \dot{q})\dot{q} + F(\cdot, q, \dot{q})$ uniformly. Finally, we will check that $M(q_\lambda)\ddot{q_\lambda}$ weakly converges to $M(q)\ddot{q}$ in the $\sigma(L^\infty, L^1)$ sense. Note that from Assumption 4.2.1 $(\mathcal{M}_1)$:

$$\|M(q(t))\ddot{q}(t)\| \leq \|M(q(t))\|_m\|\ddot{q}(t)\| \leq k_2\|\ddot{q}(t)\|$$

which means $M(q)\ddot{q} \in L^\infty([0,T],\mathbb{R}^n)$. Similarly, we have $M(q_\lambda)\ddot{q_\lambda} \in L^\infty([0,T],\mathbb{R}^n)$. For every $\varphi \in L^1([0,T],\mathbb{R}^n)$ we have $M(q)\varphi \in L^1([0,T],\mathbb{R}^n)$ and:

$$
\begin{aligned}
&|\int_0^T \langle M(q_\lambda(t))\ddot{q_\lambda}(t) - M(q)(t)\ddot{q}(t), \varphi(t)\rangle dt| \\
&\leq |\int_0^T \langle [M(q_\lambda(t)) - M(q(t))]\ddot{q_\lambda}(t), \varphi(t)\rangle dt| + |\int_0^T \langle M(q(t))(\ddot{q_\lambda}(t) - \ddot{q}(t)), \varphi(t)\rangle dt| \\
&\leq k_3 \int_0^T \|q_\lambda(t) - q(t)\|\|\ddot{q_\lambda}(t)\|\|\varphi(t)\|dt + |\int_0^T \langle(\ddot{q_\lambda}(t) - \ddot{q}(t)), M(q(t))\varphi(t)\rangle dt| \to 0
\end{aligned}
$$

when $\lambda \to 0$ since $\ddot{q_\lambda}$ converges weakly to $\ddot{q}$ and q_λ converges to q uniformly. Hence, $M(q_\lambda)\ddot{q_\lambda}$ converges weakly to $M(q)\ddot{q}$. Hence, we obtain $M(q_\lambda)\ddot{q_\lambda} + C(q_\lambda, \dot{q_\lambda})\dot{q_\lambda} + \nabla\mathcal{V}(q_\lambda) + F(\cdot, q_\lambda, \dot{q_\lambda})$ converges weakly to $M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla\mathcal{V}(q) + F(\cdot, q, \dot{q})$ in the $\sigma(L^\infty, L^1)$ sense. Since $T > 0$ can be chosen arbitrarily, we have proved the theorem. ■

Remark 4.3.2 *There is another way different from [8] to explain the convergence function $q(\cdot)$ is a solution of* (4.9) *which bases on the strong-weak closed graph property of maximal monotone operator (see [19]). Indeed, let $A := \partial\Phi$, then A is maximal monotone. Define the operator $\mathcal{A}$ from $L^\infty([0,T],\mathbb{R}^n) \subset L^1([0,T],\mathbb{R}^n)$ to its dual space $L^\infty([0,T],\mathbb{R}^n)$ by $(\mathcal{A})x(t) = A(x(t))$ almost everywhere. It is easy to check that $\mathcal{A}$ is monotone: if $u_i(\cdot) \in \mathcal{A}v_i(\cdot)$ for $i = 1,2$ then:*

$$\int_0^T \langle u_1(t) - u_2(t), v_1(t) - v_2(t)\rangle dt \geq 0. \tag{4.29}$$

Since A is maximal monotone, the mapping $J_I = (I + A)^{-1}$ is Lipschitz. Hence for each $y(\cdot) \in L^\infty[0,T],\mathbb{R}^n)$, the function $x(\cdot)$ defined by:

$$x(t) = J_I y(t) \text{ a.e.} \tag{4.30}$$

is measurable. Furthermore:

$$\|x(t) - J_I y(0)\| = \|J_I y(t) - J_I y(0)\| \leq \|y(t)\|,$$

which implies that $x(\cdot) \in L^\infty([0,T],\mathbb{R}^n) \subset L^1([0,T],\mathbb{R}^n)$ and satisfies $y(\cdot) \in x(\cdot) + \mathcal{A}x(\cdot)$. Hence, $\mathcal{A}$ is maximal monotone. Since $\dot{q_\lambda}$ converges uniformly to $\dot{q}$ and $M(q_\lambda)\ddot{q_\lambda} + C(q_\lambda, \dot{q_\lambda})\dot{q_\lambda} + \nabla\mathcal{V}(q_\lambda) + F(\cdot, q_\lambda, \dot{q_\lambda})$ converges weakly to $M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla\mathcal{V}(q) + F(\cdot,q,\dot{q})$ in $\sigma(L^\infty, L^1)$, we have:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla\mathcal{V}(q) + F(\cdot,q,\dot{q}) \in \mathcal{A}(\dot{q})$$

or equivalently for a.e. $t \geq 0$:

$$M(q(t))\ddot{q}(t) + C(q(t),\dot{q}(t))\dot{q}(t) + \nabla\mathcal{V}(q(t)) + F(t,q(t),\dot{q}(t)) \in \partial\Phi(\dot{q}(t)).$$

4.4 Uniqueness of Solutions

Let us start with a result which relies on the assumption that the matrix $M^{-1}(q)$ does not destroy the monotonicity of the operator $\partial\Phi(\cdot)$. Since the matrix M is not independent of q, the change of variables used in [3, 10, 33, 35, 36] does not work anymore. It is noted that this assumption is only required to be hold locally, like some kinds of local hypo-monotonicity.

Theorem 4.4.1 *Let Assumptions* 4.2.1, 4.2.2, 4.2.3 *hold. Moreover, assume that for all* $m, p \in \mathbb{R}^n$, *there exist* $\varepsilon > 0, \gamma > 0$ *such that for all* $p_1, p_2 \in \mathbb{B}_\varepsilon(p)$, *we have:*

$$\langle M^{-1}(m)(p_1^* - p_2^*), p_1 - p_2\rangle \geq -\gamma\|p_1 - p_2\|^2,\ \forall p_1^* \in \partial\Phi(p_1), \forall p_2^* \in \partial\Phi(p_2). \tag{4.31}$$

Then (4.9) has a unique solution in the sense of Theorem 4.3.1.

Proof. For arbitrary $T > 0$, suppose that $(q_1(\cdot), \dot{q}_1(\cdot))$, $(q_2(\cdot), \dot{q}_1(\cdot))$ are two solutions of (4.9) on $[0, T]$ with the same initial conditions. Let $z_i := \left(q_i^T\ \dot{q}_i^T\right)^T$ for $i = 1, 2$, then $z_1(\cdot)$ and $z_2(\cdot)$ are two solutions of the differential inclusion:

$$\dot{x}(t) + A(t, x(t)) \in -B(x(t)), \tag{4.32}$$

where $x = (x_1^T\ x_2^T)^T$, $B(x) := \begin{pmatrix} 0 \\ M^{-1}(x_1)\partial\Phi(x_2) \end{pmatrix}$ and

$$A(t, x) := \begin{pmatrix} -x_2 \\ M^{-1}(x_1)[\nabla\mathcal{V}(x_1) + C(x_1, x_2)x_2 + F(t, x_1, x_2)] \end{pmatrix}.$$

As we know, $A(t, x)$ is continuous in t and uniformly locally Lipschitz in x. We now prove that $B(\cdot)$ is locally hypomonotone. Indeed, for all $m, p \in \mathbb{R}^{2n}, m = (m_1^T\ m_2^T)^T, p = (p_1^T\ p_2^T)^T, m_2^* \in \partial\Phi(m_2), p_2^* \in \partial\Phi(p_2)$ we have:

$$\begin{aligned}
&\langle M^{-1}(m_1)m_2^* - M^{-1}(p_1)p_2^*, m_2 - p_2\rangle \\
&= \langle M^{-1}(m_1)(m_2^* - p_2^*), m_2 - p_2\rangle + \langle (M^{-1}(m_1) - M^{-1}(p_1))p_2^*, m_2 - p_2\rangle \\
&\geq -\gamma\|m_2 - p_2\|^2 - k_5\|p_2^*\|\|m_1 - p_1\|\|m_2 - p_2\|,
\end{aligned}$$

where (4.31) has been used to obtain the first inequality. Therefore $B(\cdot)$ is locally hypomonotone due to the boundedness of $\partial\Phi(\cdot)$ on bounded sets. Then, there exists a $\sigma > 0$ such that $A(t, \cdot)$ is uniformly Lipschitz with constant $l_1 > 0$ and $B(\cdot)$ is hypomonotone with constant $l_2 > 0$ in $\mathbb{B}_\sigma(z_0)$, where $z_0 = (q_0^T\ \dot{q}_0^T)^T$. Due to the continuity of $z_1(\cdot), z_2(\cdot)$, there exists a T_0 such that $0 < T_0 < T$ and $z_1(t), z_2(t) \in \mathbb{B}_\sigma(z_0)$ for all $t \in [0, T_0]$. For almost all $t \in [0, T_0]$, we have:

$$\langle \dot{z}_1(t) - \dot{z}_2(t) + A(t, z_1(t)) - A(t, z_2(t)), z_1(t) - z_2(t)\rangle \leq l_2\|z_1(t) - z_2(t)\|^2,$$

which implies:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|z_1(t) - z_2(t)\|^2 &\leq \langle A(t, z_2(t)) - A(t, z_1(t)), z_1(t) - z_2(t)\rangle + l_2\|z_1(t) - z_2(t)\|^2 \\
&\leq (l_1 + l_2)\|z_1(t) - z_2(t)\|^2 = l\|z_1(t) - z_2(t)\|^2,
\end{aligned}$$

where $l = l_1 + l_2$. By Gronwall's inequality, we have $\|z_1(t) - z_2(t)\|^2 \leq 0$ for all $t \in [0, T_0]$ or, $z_1(t) = z_2(t)$ $\forall t \in [0, T_0]$. Now, we suppose there exists $t_1 \in [0, T)$ such that $z_1(t_1) \neq z_2(t_1)$. Let

$$E := \{t \in [0, t_1] : z_1(t) \neq z_2(t)\}.$$

Since $t_1 \in E$ and E is bounded from below, there exists $\alpha = \inf E$ where $\alpha \in (t_0, t_1]$ and for all $t \in [t_0, \alpha)$: $z_1(t) = z_2(t)$. By the continuity of $z_1(\cdot)$ and $z_2(\cdot)$, we have $z_1(\alpha) = z_2(\alpha)$ which implies that $\alpha < t_1$. With the same argument as above, there exists a neighborhood of α such that $z_1(\cdot) \equiv z_2(\cdot)$. This is a contradiction. So $z_1(\cdot) \equiv z_2(\cdot)$ on $[0, T]$ and hence, $q_1(\cdot) \equiv q_2(\cdot)$ on $[0, T]$. ■

The following propositions give some cases where $M(q)$ and $\Phi(\cdot)$ are such that (4.31) is satisfied.

Proposition 4.4.1 *Suppose that for each $q \in \mathbb{R}^n$, $M(q)$ is a positive definite diagonal matrix, and $\Phi(q) = \Phi_1(q_1) + \ldots + \Phi_n(q_n)$ where $\Phi_i : \mathbb{R} \to \mathbb{R}$ is a convex function, $i = 1, \ldots, n$. Then* (4.31) *holds.*

Proof. Let $x, y, z \in \mathbb{R}^n$ and $y^* \in \partial\Phi(y), z^* \in \partial\Phi(z)$. Suppose that $M^{-1}(x) = \text{diag}(k_1(x), \ldots, k_n(x))$, where $k_i(x) > 0, i = 1, 2, \ldots, n$. Then, we obtain:

$$\begin{aligned}&\langle M^{-1}(x)(y^* - z^*), y - z\rangle \\ &= k_1(x)(y_1^* - z_1^*).(y_1 - z_1) + \ldots + k_n(x)(y_n^* - z_n^*).(y_n - z_n) \geq 0,\end{aligned}$$

where $y_i^* \in \partial\Phi_i(y_i), z_i^* \in \partial\Phi_i(z_i), i = 1, 2, \ldots, n$. ■

Proposition 4.4.2 *Suppose that for each $q = (q_1\ q_2\ \ldots q_n)^T \in \mathbb{R}^n$:*
1) $M(q) = \begin{pmatrix} M_1(q) & 0 \\ 0 & M_2(q) \end{pmatrix}$ where $M_1(q) \in \mathbb{R}^{m\times m}$, and $M_2(q) \in \mathbb{R}^{p\times p}$ is a positive definite diagonal matrix, $n = m + p$.
2) $\Phi(q) = \Phi_{m+1}(q_{m+1}) + \ldots + \Phi_n(q_n)$ where $\Phi_i : \mathbb{R} \to \mathbb{R}$ is a convex function, $i = m + 1, \ldots, n$.
Then the condition (4.31) *is satisfied.*

Proof. Let $x, y, z \in \mathbb{R}^n$ and $y^* \in \partial\Phi(y), z^* \in \partial\Phi(z)$. Assume that $M_2^{-1}(x) = \text{diag}(k_{m+1}(x), \ldots, k_n(x))$, where $k_i(x) > 0, i = m + 1, .., n$. Then, we have:

$$\begin{aligned}&\langle M^{-1}(x)(y^* - z^*), y - z\rangle = k_{m+1}(x)(y_{m+1}^* - z_{m+1}^*).(y_{m+1} - z_{m+1}) + \\ &\ldots + k_n(x)(y_n^* - z_n^*).(y_n - z_n) \geq 0,\end{aligned}$$

where $y_i^* \in \partial\Phi_i(y_i), z_i^* \in \partial\Phi_i(z_i), i = m + 1, \ldots, n$. ■

Remark 4.4.1 *i) We also have the uniqueness result if we replace assumption 2) in the Proposition* 4.4.2 *by:*
2') $\Phi(q) = \Psi(q_1, \ldots, q_m) + \Phi_{m+1}(q_{m+1}) + \ldots + \Phi_n(q_n)$ where $\Phi_i : \mathbb{R} \to \mathbb{R}$ is a convex function, $i = m + 1, \ldots, n$, $\Psi \in \mathcal{C}^1(\mathbb{R}^m; \mathbb{R})$ is convex and $\nabla\Psi$ is locally Lipschitz.

Indeed, we can put the term $\nabla\Psi(\cdot)$ into $A(\cdot)$ and the new function $A(\cdot)$ is still locally Lipschitz (in x).

ii) If M is a constant matrix then the change of coordinates that has been used in [10, 33, 35, 36] can be used to prove the uniqueness by setting $z = M^{\frac{1}{2}}q$ and $\varphi(\cdot) = \Phi \circ M^{-\frac{1}{2}}(\cdot)$.

Let us note that other results for the uniqueness of solutions exist, like Filippov's criterion for codimension one attractive surfaces [55]. Filippov's criterion may apply to (4.9) for specific choices of $\Phi(\cdot)$, for example, $\Phi(\cdot) = |D^T \circ \cdot|$ as in the following Proposition. This result shows the link between our development and the sliding mode in control.

Proposition 4.4.3 *If for all $y \in \mathbb{R}^n$, $\Phi(y) = |D^T y|$ where $D \neq 0$ is a vector in $\mathbb{R}^n$, then for any initial condition, the solution of* (4.9) *in the sense of Theorem* 4.3.1 *is unique.*

Proof. It is easy to compute that $\partial\Phi(y) = D\text{Sign}(D^T y)$. Let $h : \mathbb{R}^{2n} \to \mathbb{R}, h(x) = D^T x_2$ where $x = (x_1^T \ x_2^T)^T$ and

$$\Sigma := \{x \in \mathbb{R}^{2n} : h(x) = 0\},\ S^- := \{x \in \mathbb{R}^{2n} : h(x) < 0\},\ S^+ := \{x \in \mathbb{R}^{2n} : h(x) > 0\}. \tag{4.33}$$

Then (4.9) is equivalent to the first order system:

$$\dot{x} \in \Gamma(x,t) := \begin{cases} \Gamma^-(x,t), \text{ if } x \in S^-, \\ \\ \overline{\text{co}}\{\Gamma^-(x,t), \Gamma^+(x,t)\} \text{ if } x \in \Sigma, \\ \\ \Gamma^+(x,t) \text{ if } x \in S^+, \end{cases} \tag{4.34}$$

where $\Gamma^-, \Gamma^+ : \mathbb{R}^{2n+1} \to \mathbb{R}$ are defined by:

$$\Gamma^-(x,t) := \left(\begin{array}{c} x_2 \\ -M^{-1}(x_1)[\nabla\mathcal{V}(x_1) + C(x_1,x_2)x_2 + F(t,x_1,x_2) - D] \end{array} \right),$$

and

$$\Gamma^+(x,t) := \left(\begin{array}{c} x_2 \\ -M^{-1}(x_1)[\nabla\mathcal{V}(x_1) + C(x_1,x_2)x_2 + F(t,x_1,x_2) + D] \end{array} \right).$$

Note that for such type of system, our solutions (in the sense of Theorem 4.3.1) and Filippov's solutions coincide. Indeed, if $(y(\cdot)\ \dot{y}(\cdot))$ is a solution of (5.7) in the sense of Theorem 4.3.1, then $(y(\cdot)\ \dot{y}(\cdot))$ is absolutely continuous and satisfies (5.7) for *a.e* $t \geq 0$. Therefore, it is also a Filippov's solutions. Inversely, let $(y(\cdot)\ \dot{y}(\cdot))$ is a Filippov's solution of (5.7). Then $y \in \mathcal{C}^1([0,+\infty), \mathbb{R}^n)$ and $(y(\cdot)\ \dot{y}(\cdot))$ is bounded on $[0,T]$ for all $T > 0$. From (5.7), we have $\ddot{y}(\cdot)$ is bounded *a.e.* on $[0,T]$. Hence $(y(\cdot)\ \dot{y}(\cdot))$ is a solution of (5.7) in the sense of Theorem 4.3.1.

Let $n := \nabla h = \left(\begin{array}{c} 0 \\ D \end{array} \right)$ and

$$\begin{array}{rcl} \Gamma_n^-(x,t) & := & \langle \Gamma^-(x,t), n\rangle = \langle -M^{-1}(x_1)[\nabla\mathcal{V}(x_1) + C(x_1,x_2)x_2 + F(t,x_1,x_2)], D\rangle \\ & + & \langle M^{-1}(x_1)D, D\rangle, \end{array}$$

$$\begin{aligned}\Gamma_n^+(x,t) &:= \langle \Gamma^-(x,t), n\rangle = \langle -M^{-1}(x_1)[\nabla\mathcal{V}(x_1) + C(x_1,x_2)x_2 + F(t,x_1,x_2)], D\rangle \\ &- \langle M^{-1}(x_1)D, D\rangle.\end{aligned}$$

Then:

$$\Gamma_n^-(x,t) - \Gamma_n^+(x,t) = 2\langle M^{-1}(x_1)D, D\rangle > 0. \tag{4.35}$$

and the uniqueness of solutions follows by using Theorem 2.3.6. ■

Remark 4.4.2 *Let us note that in general for switching surfaces of codimension* ≥ 2, *Filippov's differential inclusions do not have unique solutions, even if the switching surface is attractive. Example of non-uniqueness with codimension* 2 *attractive surface can be found in [72]. So our study of uniqueness makes sense, if one is interested in this property.*

4.5 Stability Analysis

In this section, the Lyapunov stability of equilibria and attractivity of the equilibrium set of (4.9) are studied. It is remarkable that from here, we do not need the uniqueness of solutions. We make the following assumption through this section and the next one:

Assumption 4.5.1

$$F(t,x_1,x_2) \equiv F(x_1,x_2),\ (t,x_1,x_2) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n. \tag{4.36}$$

The equilibrium set $\mathcal{W}$ of (4.9) is given by:

$$\mathcal{W} := \{s \in \mathbb{R}^n :\ \nabla\mathcal{V}(s) + F(s,0) \in -\partial\Phi(0)\}. \tag{4.37}$$

Assumption 4.5.2

$$\nabla\mathcal{V}(0) + F(0,0) \in -\partial\Phi(0). \tag{4.38}$$

Then, we have $0 \in \mathcal{W}$. We reduce (4.9) into the first order differential inclusion:

$$\dot{x}(t) \in \mathcal{F}(x(t)), \tag{4.39}$$

where $x = (x_1^T\ x_2^T)^T$ and:

$$\mathcal{F}(x) := \begin{pmatrix} x_2 \\ -M^{-1}(x_1)[\nabla\mathcal{V}(x_1) + C(x_1,x_2)x_2 + \partial\Phi(x_2) + F(x_1,x_2)] \end{pmatrix}. \tag{4.40}$$

It is easy to check that:

$$\mathcal{Y} = \mathcal{W}\times\{0\}, \tag{4.41}$$

where $\mathcal{Y}$ is the set of stationary solutions of (4.39). The next result is an extension of the Lagrange - Dirichlet theorem of mechanics (also called the Lejeune-Dirichlet theorem), where $\mathcal{V}(q)$ is the potential energy while $\partial\Phi(\dot{q})$ is the dissipation term.

Assumption 4.5.3 *(a) There exists an $\alpha \geq 0$ such that:*

$$\Phi(\cdot) \geq \alpha\|\cdot\| \qquad \text{and} \qquad \sup_{(x_1,x_2)\in\times\mathbb{R}^n\times\mathbb{R}^n} \|F(x_1,x_2)\| \leq \alpha. \tag{4.42}$$

(b) There exist $\alpha > \beta \geq 0$ such that:

$$\Phi(\cdot) \geq \alpha\|\cdot\| \qquad \text{and} \qquad \sup_{(x_1,x_2)\in\times\mathbb{R}^n\times\mathbb{R}^n} \|F(x_1,x_2)\| \leq \beta. \tag{4.43}$$

Remark 4.5.1 *If $F \equiv 0$, then $\alpha = 0$ and Assumption 4.5.3 (a) holds. Note that the condition $0 \in \mathrm{int}(\partial\Phi(0))$ is equivalent to $\Phi(\cdot) \geq \alpha\|\cdot\|$ for some $\alpha > 0$ (see [8]).*

Theorem 4.5.1 *(Stability) Let the assumptions of Theorem 4.3.1 and Assumption 4.5.1, 4.5.2, 4.5.3 (a) hold. Suppose that $\mathcal{V}(\cdot)$ is locally positive definite. Then the origin of the system (4.39) is stable.*

Proof. Consider the Lyapunov function $V : \mathbb{R}^{2n} \to \mathbb{R}$ defined by:

$$V(x) := \frac{1}{2}\langle M(x_1)x_2, x_2\rangle + \mathcal{V}(x_1). \tag{4.44}$$

Then $V(\cdot)$ is differentiable and locally positive definite. We prove that the derivative of $V(\cdot)$ along the trajectories of the system is non-positive. Indeed, for almost all $t \geq 0$, we have:

$$\begin{aligned}
&\dot{V}(x(t)) = \frac{d}{dt}(V \circ x(t))\\
&= \frac{1}{2}\langle\frac{d}{dt}(M(x_1(t)))x_2(t), x_2(t)\rangle + \langle x_2(t), M(x_1(t))\dot{x}_2(t)\rangle + \langle\nabla\mathcal{V}(x_1(t)), \dot{x}_1(t)\rangle\\
&= \frac{1}{2}\langle\frac{d}{dt}(M(x_1(t)))x_2(t), x_2(t)\rangle - \langle C(x_1(t), x_2(t))x_2(t) + \nabla\mathcal{V}(x_1(t)) + F(x_1(t), x_2(t))\\
&+\omega(t), x_2(t)\rangle + \langle\nabla\mathcal{V}(x_1(t)), x_2(t)\rangle\\
&= \Big\langle\frac{1}{2}[\frac{d}{dt}(M(x_1(t))) - 2C(x_1(t), x_2(t))]x_2(t), x_2(t)\Big\rangle - \langle\omega(t) + F(x_1(t), x_2(t)), x_2(t)\rangle\\
&= \langle\omega(t) + F(x_1(t), x_2(t)), -x_2(t)\rangle \leq \Phi(0) - \Phi(x_2(t)) + \alpha\|x_2(t)\| \leq 0,
\end{aligned}$$

for some $\omega(t) \in \partial\Phi(x_2(t))$ and where Assumption 4.5.3 (a) and the skew-symmetry property $(\mathcal{C}_2)$ of Assumption 3.2 have been used. Next, we prove that $x = 0$ is stable. Since $V(\cdot)$ is locally positive definite, there exist $h > 0$ and a strictly increasing function $\rho(\cdot) \in \mathcal{C}(\mathbb{R}^+; \mathbb{R})$ with $\rho(0) = 0$ such that:

$$V(x) \geq \rho(\|x\|) \text{ for all } x \in \mathbb{B}_h.$$

Without loss of generality, let $0 < \varepsilon < h$ and let $c = \rho(\varepsilon)$. Because $V(\cdot)$ is locally positive definite, there exists $\eta > 0$ such that $\mathbb{B}_\eta \subset \Omega_c^\circ = \{x \in \mathbb{R}^{2n} : V(x) < c\}$. Let $\delta = \min\{\varepsilon, \eta\}$. Let $x_0 \in \mathbb{B}_\delta$ and $x(t; x_0)$ be a solution of (4.39) satisfying the initial condition $x(0; x_0) = x_0$. Suppose that there exists $t_1 \geq t_0$ such that $\|x(t_1; x_0)\| \geq \varepsilon$. Since $x(\cdot; x_0)$ is continuous, we may find some t^* satisfying: $\|x(t^*; x_0)\| = \varepsilon$. Then:

$$V(x(t^*; x_0)) \geq \rho(\|x(t^*; x_0)\|) = \rho(\varepsilon).$$

On the other hand, $V(\cdot)$ is decreasing along the trajectory, we have:

$$V(x(t^*;x_0)) \leq V(x_0) < c = \rho(\varepsilon).$$

Our proof is finished by contradiction. ■
In the sequel of this section, we will generalize the Krasovskii-LaSalle invariance principle to prove the asymptotic stability of the equilibrium set. Firstly, we recall some definitions and properties. Let $x_0 \in \mathbb{R}^{2n}$ and $\psi(t;x_0)$ be a solution of (4.39), denote the *orbit* of ψ by:

$$\gamma(\psi) := \{\psi(t;x_0) :\ t \geq 0\} \subset \mathbb{R}^{2n},$$

and the *limit set* of ψ by:

$$\Lambda(\psi) := \{p \in \mathbb{R}^{2n} :\ \exists\{t_i\}, t_i \to \infty \ \text{ as } i \to \infty \text{ and } \psi(t_i;x_0) \to p\}.$$

A set $S \in \mathbb{R}^{2n}$ is said *weakly invariant* if and only if for $x_0 \in S$, there exists a solution of (4.39) starting at x_0 contained in S. It is said *invariant* if and only if for $x_0 \in S$, all solutions of (4.39) starting at x_0 are contained in S.

Remark 4.5.2 *The following properties are classical (see [10]).*

(i) If $\gamma(\psi)$ is bounded, then $\Lambda(\psi) \neq \emptyset$ and

$$\lim_{t\to\infty} d(\psi(t;x_0), \Lambda(\psi)) = 0.$$

(ii) The set of stationary solutions $\mathcal{Y}$ is weakly invariant. In fact, if $x_0 \in \mathcal{Y}$ then the solution $\psi(t;x_0) = x_0,\ t \geq t_0$, is contained in $\mathcal{Y}$.

(iii) It is known that when $\mathcal{F}(\cdot)$ is upper semicontinuous with nonempty, convex, compact values and $\psi(t;x_0)$ is a solution of (4.39) *then its limit set $\Lambda(\psi)$ is weakly invariant [23].*

Lemma 4.5.1 *The function $\mathcal{F}(\cdot)$ defined in* (5.2) *is upper semicontinuous with nonempty, convex, compact values.*

Proof. The subdifferential $\partial\Phi(\cdot)$ is upper semicontinuous with nonempty, convex, compact values (Proposition 2.2.1). Therefore, particularly for all $x \in \mathbb{R}^{2n}$, $\mathcal{F}(x)$ is nonempty, convex and compact. It remains to prove that $\mathcal{F}(\cdot)$ is upper semicontinuous. Since the mapping

$$x \to \begin{pmatrix} x_2 \\ M^{-1}(x_1)[\nabla\mathcal{V}(x_1) + C(x_1,x_2)x_2 + F(x_1,x_2)] \end{pmatrix}$$

is continuous, it is sufficient to prove that the mapping

$$\mathcal{F}' \ :\ x \to M^{-1}(x_1)\partial\Phi(x_2)$$

is upper semicontinuous due to Proposition 2.2.4. Note that $\partial\Phi(\cdot)$ is also bounded on bounded sets, so $\mathcal{F}'(\cdot)$ is bounded in a neighborhood of each point $x \in \mathbb{R}^{2n}$. By Proposition 2.2.5, it is equivalent to prove that the graph of $\mathcal{F}'(\cdot)$ is closed. Let $(y_n) \subset \mathbb{R}^n$ and $(x_n) =$

$(x_{1n}^T \quad x_{2n}^T)^T \subset \mathbb{R}^{2n}$ be two sequences such that $y_n \to y$, $x_n \to x = (x_1^T \quad x_2^T)^T$ and $y_n \in \mathcal{F}'(x_n) = M^{-1}(x_{1n})\partial\Phi(x_{2n})$. Then for all $n \geq 1$, we have $M(x_{1n})y_n \in \partial\Phi(x_{2n})$. Since

$$\begin{aligned} \|M(x_{1n})y_n - M(x_1)y\| &\leq \|M(x_{1n})y_n - M(x_{1n})y\| + \|M(x_{1n})y - M(x_1)y\| \\ &\leq k_2\|y_n - y\| + k_3\|x_{1n} - x_1\|.\|y\| \end{aligned}$$

where k_2, k_3 are two constants defined in Assumption 4.2.1, we obtain that $M(x_{1n})y_n \to M(x_1)y$. Moreover, $x_{2n} \to x_2$ and the graph of $\partial\Phi(\cdot)$ is closed, it results that $M(x_1)y \in \partial\Phi(x_2)$, i.e. $y \in M^{-1}(x_1)\partial\Phi(x_2) = \mathcal{F}'(x)$. Therefore, the graph of $\mathcal{F}'(\cdot)$ is closed and the result follows. ■

Lemma 4.5.2 *Let the assumptions of Theorem* 4.3.1 *and Assumption* 4.5.3 (*a*) *hold. Let* Ω *be a compact invariant subset of* $\mathbb{R}^{2n}$. *Denote:*

$$\mathcal{Z}_\Omega := \{x = (x_1^T \; x_2^T)^T \in \Omega : \; \exists\, \omega \in \partial\Phi(x_2) \text{ such that } \langle \omega + F(x_1, x_2), x_2 \rangle = 0\}.$$

Let $\mathcal{M}$ *be the largest weakly invariant set in the closure of* $\mathcal{Z}_\Omega$. *For each* $x_0 \in \Omega$, *let* $\psi(t; x_0)$ *be a solution of* (4.39). *Then, we have:*

$$\lim_{t\to\infty} d(\psi(t; x_0), \mathcal{M}) = 0.$$

Proof. Since $x_0 \in \Omega$ and Ω is invariant, we have $\gamma(\psi) \subset \Omega$. Then $\gamma(\psi)$ is bounded and:

$$\lim_{t\to\infty} d(\psi(t; x_0), \Lambda(\psi)) = 0.$$

It is enough to prove that $\Lambda(\psi) \subset \bar{\mathcal{Z}}_\Omega$. Let us consider $V(\cdot)$ as in (4.44). Note that the function $V(\cdot)$ is $\mathcal{C}^1$, hence it is bounded on the compact set Ω. Furthermore, since $V(\psi(t; x_0))$ is decreasing with respect to t, there exists a number k such that $\lim_{t\to\infty} V(\psi(t; x_0)) = k$. For each $p \in \Lambda(\psi)$, there exist $\{t_i\}, t_i \to \infty$ as $i \to \infty$ and $\psi(t_i; x_0) \to p$. Then $V(p) = k$ due to the continuity of $V(\cdot)$. Hence $V(p) = k$ for all $p \in \Lambda(\psi)$. Let $z \in \Lambda(\psi)$. Since $\Lambda(\psi)$ is weakly invariant, there exists a solution $\phi(t; z)$ of (4.39) lying in $\Lambda(\psi)$. Therefore:

$$V(\phi(t; z)) = k$$

for all $t \geq 0$, which implies:

$$0 = \dot{V}(\phi(t; z)) = -\langle \omega(t) + F(\phi_1(t; z), \phi_2(t; z)), \phi_2(t; z) \rangle$$

for almost all $t \geq 0$, where $\omega(t) \in \partial\Phi(\phi_2(t; z))$ and $\phi(t) = (\phi_1^T(t) \;\; \phi_2^T(t))^T$. Hence, we have:

$$\phi(t; z) \in \mathcal{Z}_\Omega$$

for almost all $t \geq 0$. Since $\phi(\cdot; t_0, z)$ is continuous, we obtain:

$$z = \phi(0; z) \in \bar{\mathcal{Z}}_\Omega,$$

and the result follows. ■

Theorem 4.5.2 *(Attractivity) Let the assumptions of Theorem* 4.3.1 *hold. Furthermore, suppose that:*
(i) Assumption 4.5.3 *(b) holds.*
(ii) $\mathcal{V}(\cdot)$ *is radially unbounded.*
Then, for given $\psi_0 = (q_0^T\ \dot{q}_0^T)^T \in \mathbb{R}^{2n}$*, if* $\psi(t; x_0) = (q(t; q_0, \dot{q}_0)^T\ \dot{q}(t; q_0, \dot{q}_0)^T)^T$ *is a solution of* (4.39)*, we have:*

$$\lim_{t\to\infty} d(\psi(t; x_0), \mathcal{Y}) = 0, \tag{4.45}$$

or equivalently:

$$\lim_{t\to\infty} d(q(t; q_0, \dot{q}_0), \mathcal{W}) = 0 \text{ and } \lim_{t\to\infty} \dot{q}(t; q_0, \dot{q}_0) = 0, \tag{4.46}$$

where $\mathcal{W}$ *and* $\mathcal{Y}$ *are defined in* (4.37) *and* (4.41) *respectively.*

Proof. Consider the same Lyapunov function as in Theorem 4.5.1, then:
1) $V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$.
2) $V(\cdot)$ is bounded from below.
3) $V(\cdot)$ is radially unbounded.
4) $V(\cdot)$ is decreasing along the trajectories since the orbital derivative:

$$\begin{aligned}\dot{V}(x(t)) &= \langle -x_2(t), \omega(t) + F(x_1(t), x_2(t))\rangle \le \Phi(0) - \Phi(x_2(t)) + \beta\|x_2(t)\| \\ &\le -(\alpha - \beta)\|x_2(t)\| \le 0,\end{aligned}$$

where $\omega(t) \in \partial\Phi(x_2(t))$ due to the assumption (i). We have $\dot{V}(x) = 0$ if and only if $x_2 = 0$. For given $x_0 = (q_0^T\ \dot{q}_0^T)^T \in \mathbb{R}^{2n}$, let $\Omega = \{x \in \mathbb{R}^{2n} :\ V(x) \le V(x_0)\}$. Then Ω is a non-empty, compact subset of $\mathbb{R}^{2n}$. Moreover, for $z \in \Omega$ and $\phi(t; z)$ is a solution of (4.39), we have for all $t \ge 0, V(\phi(t; z)) \le V(\phi(0; z)) = V(z) \le V(x_0)$ which implies $\phi(t; z) \in \Omega$ for all $t \ge 0$. Hence Ω is invariant and $x_0 \in \Omega$. Note that:

$$\begin{aligned}\mathcal{Z}_\Omega &= \{x = (x_1^T\ x_2^T)^T \in \Omega :\ \exists\, \omega \in \partial\Phi(x_2) \text{ such that } \langle \omega + F(x_1, x_2), x_2\rangle = 0\} \\ &= \Omega \cap (\mathbb{R}^n \times \{0\}),\end{aligned}$$

where the last equality is obtained using assumption (i) of the Theorem. In fact, we have: $0 = \langle \omega + F(x_1, x_2), x_2\rangle \ge \Phi(x_2) - \Phi(0) + \langle F(x_1, x_2), x_2\rangle \ge (\alpha - \beta)\|x_2\| \ge 0$, which implies that $x_2 = 0$. Let $\mathcal{M}$ be the largest weakly invariant set in $\bar{\mathcal{Z}}_\Omega \equiv \mathcal{Z}_\Omega$. Then by Lemma 4.5.2 we have:

$$\lim_{t\to\infty} d(\psi(t; x_0), \mathcal{M}) = 0.$$

We recall that the set $\mathcal{Y}$ of stationary solutions of (4.39) is weakly invariant and $\mathcal{Y} = \mathcal{W}\times\{0\}$. Hence, $\mathcal{Y}_\Omega := \Omega\cap\mathcal{Y}$ is a weakly invariant subset of $\mathcal{Z}_\Omega$. We now prove that $\mathcal{Y}_\Omega \equiv \mathcal{M}$. Indeed, let $\mathcal{D}$ be a weakly invariant set in $\mathcal{Z}_\Omega$ and take $z = (z_1^T\ z_2^T)^T \in \mathcal{D}$. Then there exists a solution $\theta(t; , z)$ lying in $\mathcal{D}$ and for all most all $t \ge 0$:

$$\begin{cases} \dot{\theta}_1(t; z) = \theta_2(t; z), \\ \dot{\theta}_2(t; z) + M^{-1}(\theta_1(t; z))[C(\theta_1(t; z), \theta_2(t; z))\theta_2(t; z) + \nabla\mathcal{V}(\theta_1(t; z)) \\ +F(\theta_1(t; z), \theta_2(t; z))] \in -M^{-1}(\theta_1(t; z))\partial\Phi(\theta_2(t; z)). \end{cases} \tag{4.47}$$

Since the orbit $\gamma(\theta) \subset \mathcal{D} \subset \mathcal{Z}_\Omega$, we infer from (4.47) that $\theta_2(t;z) = 0$ for all $t \geq 0$. Hence, $z_2 = 0$, $\theta_1(t;z) \equiv \theta_1(0;z) = z_1$ and $\mathcal{V}(z_1) + F(z_1, 0) \in -\partial\Phi(0)$. This means that $z \in \mathcal{Y}_\Omega$ which leads to $\mathcal{D} \subset \mathcal{Y}_\Omega$. Therefore, $\mathcal{Y}_\Omega$ is the largest weakly invariant set in $\mathcal{Z}_\Omega$ and:

$$\lim_{t\to\infty} d(\psi(t;x_0), \mathcal{Y}_\Omega) = 0.$$

It implies that:

$$\lim_{t\to\infty} d(\psi(t;x_0), \mathcal{Y}) = 0.$$

or equivalently:

$$\lim_{t\to\infty} d(q(t;q_0,\dot{q}_0), \mathcal{W}) = 0 \text{ and } \lim_{t\to\infty} \dot{q}(t;q_0,\dot{q}_0) = 0.$$

■

Remark 4.5.3 *The set $\mathcal{Y}_\Omega$ is clearly a better estimation than $\mathcal{Y}$ that helps us predict the convergence of trajectories more accurately. From the proof of the theorem above, we can also imply that:*

$$\lim_{t\to\infty} d(q(t;q_0,\dot{q}_0), \mathcal{W}_\Omega) = 0 \text{ and } \lim_{t\to\infty} \dot{q}(t;q_0,\dot{q}_0) = 0,$$

where $\mathcal{W}_\Omega$ is defined from:

$$\mathcal{Y}_\Omega = \mathcal{W}_\Omega \times \{0\}.$$

The assumption on the radial unboundedness of $\mathcal{V}$ plays a key role for the global attractivity. However, it is not always satisfied in general. Therefore, it makes sense to consider the local attractivity of the systems without this assumption.

Lemma 4.5.3 *Let the assumptions of the Theorem* 4.5.1 *and Assumption* 4.5.3 (*b*) *hold. Then there exists a neighborhood $K \subset \mathbb{R}^{2n}$ of the origin such that for each $x_0 \in K$ and $\psi(t;x_0)$ be a solution of* (4.39), *we have:*

$$\lim_{t\to\infty} d(\psi(t;x_0), \mathcal{Y}_\Omega) = 0.$$

Proof. Consider the same Lyapunov function as in Theorem 4.5.1, then the origin of the system (4.39) is stable. Fix a compact set Ω containing the origin. Then, there exists a neighborhood $K \subset \mathbb{R}^{2n}$ of the origin such that for each $x_0 \in K$, the orbit $\gamma(\psi)$ remains in the compact set Ω. Hence, $\gamma(\psi)$ is bounded and we have:

$$\lim_{t\to\infty} d(\psi(t;x_0), \Lambda(\psi)) = 0.$$

Similarly as in the proof of Lemma 4.5.2 and Theorem 4.5.2, we have:

$$\lim_{t\to\infty} d(\psi(t;x_0), \mathcal{Y}_\Omega) = 0.$$

■

Remark 4.5.4 *From Assumption* 4.5.2, *we already have: $0 \in \mathcal{Y}_\Omega$. In some cases, we can choose Ω small enough such that $\mathcal{Y}_\Omega = \{0\}$. Then, we obtain:*

$$\lim_{t\to\infty} \psi(t;x_0) = 0.$$

4.6 Finite - Time Convergence

In this section, the finite-time convergence of trajectories towards the equilibrium point is investigated. Besides the robustness, this useful property is a crucial feature of discontinuous and sliding mode controllers systems [31, 88, 89, 110, 116, 121].

Theorem 4.6.1 *Let Assumptions* 4.2.1, 4.2.2, 4.2.3, 4.5.1, 4.5.3 (*b*) *hold and let* $(q(\cdot), \dot{q}(\cdot))$ *be a solution of the system* (4.9). *The energy function is given by* $V(q, \dot{q}) = \frac{1}{2}\langle M(q)\dot{q}, \dot{q}\rangle + \mathcal{V}(q)$. *Suppose that* $\mathcal{V}(\cdot)$ *is radially unbounded. Then:*
(i) $\dot{q} \in L^{\infty}([0, +\infty), \mathbb{R}^n)$.
(ii) $\dot{q} \in L^{1}([0, +\infty), \mathbb{R}^n)$ *and hence* $q_{\infty} = \lim_{t\to\infty} q(t)$ *exists. Furthermore, we have:* $\lim_{t\to\infty} \dot{q}(t) = 0$.
(iii) The limit point q_{∞} *satisfies:*

$$\nabla\mathcal{V}(q_{\infty}) + F(q_{\infty}, 0) \in -\partial\Phi(0), \tag{4.48}$$

i.e., q_{∞} *is an equilibrium point of* (4.9).

Proof. (i) Differentiating $V(\cdot)$ along the system's trajectories as in the proof of Theorem 4.5.1, we have for a.e. $t \geq 0$:

$$\dot{V}(q(t), \dot{q}(t)) \leq \Phi(0) - \Phi(\dot{q}(t)) + \beta\|\dot{q}(t)\| \leq -(\alpha - \beta)\|\dot{q}(t)\| \leq 0. \tag{4.49}$$

From $\frac{1}{2}k_1\|\dot{q}(t)\|^2 \leq V(q(t), \dot{q}(t)) - \mathcal{V}(q(t)) \leq V(q_0, \dot{q}_0) - \inf \mathcal{V}$, we obtain: $\dot{q} \in L^{\infty}([0, +\infty), \mathbb{R}^n)$.

(ii) Next, we integrate inequality (4.49) between 0 and t which implies:

$$(\alpha - \beta)\int_0^t \|\dot{q}(s)\|ds \leq V(q_0, \dot{q}_0) - V(q(t), \dot{q}(t)) \leq V(q_0, \dot{q}_0) - \inf \mathcal{V}.$$

Let $t \to +\infty$, it follows that $\dot{q} \in L^{1}([0, +\infty), \mathbb{R}^n)$.

From $\mathcal{V}(q(t)) \leq V(q(t), \dot{q}(t)) \leq V(q_0, \dot{q}_0) < \infty$ and the radial unboundedness of $\mathcal{V}(\cdot)$, we have $q \in L^{\infty}([0, +\infty), \mathbb{R}^n)$. Then $\nabla\mathcal{V}(q)$ is bounded due to the boundedness of $\nabla\mathcal{V}(\cdot)$ on bounded sets (see Assumptions 4.2.3). Since the map $\dot{q}(\cdot)$ is bounded and $\partial\Phi(\cdot)$ is bounded on bounded sets (see Proposition 2.2.1), we have that $\partial\Phi(\dot{q})$ is bounded. Furthermore, from $(\mathcal{C}_1)$ of Assumption 3.2, we have $\|C(q, \dot{q})\dot{q}\| \leq k_4\|\dot{q}\|^2$ which implies that $C(q, \dot{q})\dot{q}$ is also bounded. Note that, $F(\cdot, q, \dot{q})$ is also bounded. Therefore, $\ddot{q}$ is bounded and hence $\dot{q}$ is Lipschitz continuous. This combined with the fact that $\dot{q} \in L^{1}([0, +\infty), \mathbb{R}^n)$ classically implies that $\lim_{t\to\infty} \dot{q}(t) = 0$, see *e.g.* [38, §4.3].

(iii) Assume that $\nabla\mathcal{V}(q_{\infty}) + F(q_{\infty}, 0) \notin -\partial\Phi(0)$ or equivalently, $0 \notin \Sigma_0 := \{z \in \mathbb{R}^n : z \in \partial\Phi(0) + \nabla\mathcal{V}(q_{\infty}) + F(q_{\infty}, 0)\}$. Then, the convex compact set $\{0\}$ and the closed convex set Σ_0 can be separated strictly by a hyperplane (see for instance Corollary 11.4.2 [103]). It means that there exist $v \in \mathbb{R}^n$ and $k > 0$ such that:

$$\langle p + \nabla\mathcal{V}(q_{\infty}) + F(q_{\infty}, 0), v\rangle > k, \tag{4.50}$$

for all $p \in \partial\Phi(0)$. Then there exists $t' \geq 0$ such that:

$$\langle \dot{q}^*(t) + \nabla \mathcal{V}(q(t)) + F(q(t), \dot{q}(t)), v \rangle > k, \tag{4.51}$$

for all $t \geq t', \dot{q}^*(t) \in \partial\Phi(\dot{q}(t))$. Indeed, suppose the contrary, then there exist sequences (w_n), (t_n) such that $t_n \to +\infty$, $w_n \in \partial\Phi(\dot{q}(t_n))$ and

$$\langle w_n + \nabla \mathcal{V}(q(t_n)) + F(q(t_n), \dot{q}(t_n)), v \rangle \leq k,$$

for all $n \geq 1$. The sequence (w_n) is bounded since the set-valued mapping $\partial\Phi(\cdot)$ is bounded on bounded sets. Therefore, there exists $w \in \mathbb{R}^n$ and a subsequence of (w_n), still denoted by (w_n) such that $w_n \to w$. Furthermore, $\dot{q}(t_n) \to 0$ and the graph of $\partial\Phi(\cdot)$ is closed in $\mathbb{R}^n \times \mathbb{R}^n$, we obtain that $w \in \partial\Phi(0)$. In addition to the assumptions on $\nabla\mathcal{V}(\cdot)$ and $F(\cdot)$, we have:

$$\langle w + \nabla \mathcal{V}(q_\infty) + F(q_\infty, 0), v \rangle \leq k,$$

which is a contradiction with (4.50).

From (4.51), we have:

$$\langle -M(q(t))\ddot{q}(t) - C(q(t), \dot{q}(t))\dot{q}(t), v \rangle > k, \text{ for all } t \geq t', \tag{4.52}$$

$$\Rightarrow k(t - t') < \int_{t'}^{t} \langle -M(q(s))\ddot{q}(s) - C(q(s), \dot{q}(s))\dot{q}(s), v \rangle ds. \tag{4.53}$$

Note that:

$$\begin{aligned}
&\frac{d}{dt}(M(q(t))\dot{q}(t)) = M(q(t))\ddot{q}(t) + \frac{d}{dt}(M(q(t)))\dot{q}(t) \\
&= M(q(t))\ddot{q}(t) + [C(q(t), \dot{q}(t)) + C(q(t), \dot{q}(t))^T]\dot{q}(t) \\
&\Rightarrow M(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t) = \frac{d}{dt}(M(q(t))\dot{q}(t)) - C(q(t), \dot{q}(t))^T \dot{q}(t),
\end{aligned}$$

where we used $(\mathcal{C}_2)$ of Assumption 3.2. Therefore, we have:

$$\begin{aligned}
&\left| \int_{t'}^{t} \langle -M(q(s))\ddot{q}(s) - C(q(s), \dot{q}(s))\dot{q}(s), v \rangle ds \right| \\
&= \left| \langle M(q(t'))\dot{q}(t') - M(q(t))\dot{q}(t), v \rangle + \int_{t'}^{t} \langle C(q(s), \dot{q}(s))^T \dot{q}(s), v \rangle ds \right| \\
&\leq \|M(q(t'))\|_m \|\dot{q}(t')\| \|v\| + \|M(q(t))\|_m \|\dot{q}(t)\| \|v\| + k_4 \|v\| \int_{t'}^{t} \|\dot{q}(s)\|^2 ds \\
&\leq k_2 \|\dot{q}(t')\| \|v\| + k_2 \|\dot{q}(t)\| \|v\| + k_7 \int_{t'}^{t} \|\dot{q}(s)\| ds < K,
\end{aligned}$$

where k_7, K are constants, due to the facts that $\dot{q} \in L^\infty([0,+\infty),\mathbb{R}^n)$ and $\dot{q} \in L^1([0,+\infty),\mathbb{R}^n)$. Then, using (4.53), we obtain for all $t \geq t'$:

$$k(t-t') \leq K,$$

which is a contradiction. Hence, we have $\nabla\mathcal{V}(q_\infty) + F(q_\infty, 0) \in -\partial\Phi(0)$. ■

Remark 4.6.1 *The condition on the radial unboundedness of $\mathcal{V}$ is used only for proving the bounded of $q(\cdot)$. In the following theorem, we give a sufficient condition ensuring that each trajectory of the system converges in finite time to an equilibrium point solution of* (4.48).

Theorem 4.6.2 *(Finite-time convergence) Let the assumptions of Theorem* 4.6.1 *hold. Moreover, assume that $-\nabla\mathcal{V}(q_\infty) - F(q_\infty,0) \notin \mathrm{bd}(\partial\Phi(0))$. Then there exists $t_f < \infty$ ($t_f \geq 0$) such that $q(t) = q_\infty$ for every $t \geq t_f$.*

Proof. From $-\nabla\mathcal{V}(q_\infty) - F(q_\infty,0) \in \partial\Phi(0)$ and $-\nabla\mathcal{V}(q_\infty) - F(q_\infty,0) \notin \mathrm{bd}(\partial\Phi(0))$, we have $-\nabla\mathcal{V}(q_\infty) - F(q_\infty,0) \in \mathrm{int}(\partial\Phi(0))$. Let $2\varepsilon = d(-\nabla\mathcal{V}(q_\infty) - F(q_\infty,0), \mathrm{bd}(\partial\Phi(0)))$. Combining with $\lim_{t\to+\infty}\{\nabla\mathcal{V}(q(t)) + F(q(t),\dot{q}(t))\} = \nabla\mathcal{V}(q_\infty) + F(q_\infty,0)$, there exists a fixed $t_1 \geq t_0$ such that for every $t \geq t_1$, we have:

$$-\nabla\mathcal{V}(q(t)) - F(q(t),\dot{q}(t)) \in \mathbb{B}_\epsilon(-\nabla\mathcal{V}(q_\infty) - F(q_\infty,0))$$

and hence:

$$-\nabla\mathcal{V}(q(t)) - F(q(t),\dot{q}(t)) + \mathbb{B}_\epsilon \in \partial\Phi(0).$$

This means that for all $t \geq t_1$ and for all $u \in \mathbb{B}_\epsilon$, we have equivalently:

$$\Phi(\dot{q}(t)) \geq \langle -\nabla\mathcal{V}(q(t)) - F(q(t),\dot{q}(t)) + u, \dot{q}(t)\rangle,$$

which implies:

$$\Phi(\dot{q}(t)) \geq \langle -\nabla\mathcal{V}(q(t)) - F(q(t),\dot{q}(t)), \dot{q}(t)\rangle + \epsilon\|\dot{q}(t)\| \text{ for all } t \geq t_1.$$

We have $\dot{V}(q(t),\dot{q}(t)) \leq -\Phi(\dot{q}(t)) - \langle F(q(t),\dot{q}(t)), \dot{q}(t)\rangle$ for almost all $t \geq t_0$ (see the proof of Theorem 4.5.1). Therefore:

$$\frac{1}{2}\frac{d}{dt}\langle M(q(t))\dot{q}(t),\dot{q}(t)\rangle + \langle\nabla\mathcal{V}(q(t)),\dot{q}(t)\rangle + \langle F(q(t),\dot{q}(t)),\dot{q}(t)\rangle + \Phi(\dot{q}(t)) \leq 0,$$

and hence for almost all $t \geq t_1$:

$$\frac{d}{dt}\langle M(q(t))\dot{q}(t),\dot{q}(t)\rangle + 2\epsilon\|\dot{q}(t)\| \leq 0. \tag{4.54}$$

Let $c(t) := \langle M(q(t))\dot{q}(t),\dot{q}(t)\rangle \leq k_2\|\dot{q}(t)\|^2 \Rightarrow \|\dot{q}(t)\| \geq \sqrt{\frac{c(t)}{k_2}}$, where k_2 is in $(\mathcal{M}_1)$ of Assumption 3.1. Therefore:

$$\dot{c}(t) + \alpha\sqrt{c(t)} \leq 0 \text{ for a.e. } t \geq t_1. \tag{4.55}$$

where $\alpha := 2\epsilon/\sqrt{k_2}$. Assume that for every $t \geq t_1, c(t) > 0$. Dividing (4.55) by $\sqrt{c(t)}$ and integrating on $[t_1, t]$, we have:

$$\sqrt{c(t)} - \sqrt{c(t_1)} \leq -\alpha(t - t_1) \text{ for all } t \geq t_1, \tag{4.56}$$

which is a contradiction. So there exists $t_f \geq t_1$ such that $c(t_f) = 0$. Note that $\dot{c}(t) \leq 0$ for almost all $t \geq t_1$, so $c(t) \leq c(t_f)$ for all $t \geq t_f$ which implies $c(t) = 0$ for all $t \geq t_f$. Therefore, $\dot{q}(t) = 0$ for all $t \geq t_f$, *i.e.* $q(t) = q_\infty$ for every $t \geq t_f$. ■

Proposition 4.6.1 *Let assumptions of Theorem* 4.6.2 *hold. Let t_f to be the first time instant such that $\dot{q}(t) = 0, q(t) = q_\infty$ for every $t \geq t_f$ and t_1 be the first time instant such that $-\nabla\mathcal{V}(q(t)) - F(q(t), \dot{q}(t))$ lies in the ball $\mathbb{B}_\varepsilon(-\nabla\mathcal{V}(q_\infty) - F(q_\infty, 0)$ and stays there for all $t \geq t_1$. Then:*

$$t_f \leq t_1 + \frac{\sqrt{k_2}\sqrt{c(t_1)}}{d(-\nabla\mathcal{V}(q_\infty) - F(q_\infty, 0), \mathrm{bd}(\partial\Phi(0)))}. \tag{4.57}$$

Proof. If $c(t_1) = 0$: the proof is trivial since $t_f = t_1$. If $c(t_1) > 0$, we must have $c(t) > 0$ for all $t \in [t_1, t_f)$ and $c(t_f) = 0$. From (4.56) we have:

$$-\sqrt{c(t_1)} = \sqrt{c(t_f)} - \sqrt{c(t_1)} \leq -\alpha(t_f - t_1), \tag{4.58}$$

which implies:

$$t_f \leq t_1 + \frac{\sqrt{c(t_1)}}{\alpha} = t_1 + \frac{\sqrt{k_2}\sqrt{c(t_1)}}{d(-\nabla\mathcal{V}(q_\infty) - F(q_\infty, 0), \mathrm{bd}(\partial\Phi(0)))}. \tag{4.59}$$

■

It is easy to check that the settling-time point t_f is inversely proportional to the distance d between $-\nabla\mathcal{V}(q_\infty) - F(q_\infty, 0)$ and the boundary of $\partial\Phi(0))$ since $c(t_1)$ can be proved to be bounded. When d tends to 0, t_f may reach to infinity. The distance function in the denominator in the right-hand-side of (4.57) depends on the control functions, consequently on the control gains. It may therefore be tuned to adjust the settling-time t_f, a property desirable in practice. Before t_1, it is possible to have some time instances at which $\dot{q}(\cdot)$ is zero. But, we can prove that these points must be isolated with additional assumptions.

Proposition 4.6.2 *Let assumptions of Theorem* 4.6.2 *hold. Suppose that $t' < t_f$ satisfies $\dot{q}(t') = 0$, where t_f is defined in Proposition* 4.6.1. *Then $-\nabla\mathcal{V}(q(t')) - F(q(t'), 0) \notin \mathrm{int}(\partial\Phi(0))$. If $-\nabla\mathcal{V}(q(t')) - F(q(t'), 0) \notin \mathrm{bd}(\partial\Phi(0))$ then t' is an isolated point in the set $\{s \in \mathbb{R}^+ : \dot{q}(s) = 0\}$.*

Proof. Suppose that $-\nabla\mathcal{V}(q(t')) - F(q(t'), 0) \in \mathrm{int}(\partial\Phi(0))$. Similarly as in the proof of Theorem 4.6.2, due to the continuity of $\mathcal{V} \circ q(\cdot)$ and $F(q(\cdot), \dot{q}(\cdot))$, there exist $\delta > 0, \varepsilon > 0$ such that

$$-\nabla\mathcal{V}(q(t)) - F(q(t), \dot{q}(t)) + \mathbb{B}_\epsilon \in \partial\Phi(0).$$

for all $t \in (t' - \delta, t' + \delta)$. Then we also have:

$$\dot{c}(t) + \alpha\sqrt{c(t)} \leq 0 \text{ for a.e. } t \in (t' - \delta, t' + \delta), \tag{4.60}$$

where $c(t)$ and α are defined in the proof of Theorem 4.6.2. Then, $c(t) \le c(t') = 0$ for all $t \in [t', t'+\delta)$. Hence, for all $t \in [t', t'+\delta)$, we have $\dot{q}(t) = 0$ and $q(t) = q(t')$. Let $t_{\max} = \sup\{\tau > t' : \dot{q}(t) = 0 \; \forall t \in [t', \tau)\}$. Assume that $t_{\max} < +\infty$. Since $q(\cdot), \dot{q}(\cdot)$ are continuous, we have $\dot{q}(t_{\max}) = 0$ and $q(t_{\max}) = q(t')$. Repeat the argument above for $t_{\max}$ instead of t', we obtain a contradiction with the definition of $t_{\max}$. So, we must have $t_{\max} = +\infty$ but it is impossible since $t' < t_f$. Therefore, $-\mathcal{V}(q(t')) - F(q(t'), 0) \notin \mathrm{int}(\partial\Phi(0))$.

If $-\nabla\mathcal{V}(q(t')) - F(q(t'), 0) \notin \mathrm{bd}(\partial\Phi(0))$, we have $-\mathcal{V}(q(t')) - F(q(t'), 0) \notin \partial\Phi(0)$. Using a separation Theorem [103], there exist $v \in \mathbb{R}^n$ and $k > 0$ satisfying:

$$\langle \nabla\mathcal{V}(q(t')) + F(q(t'), 0) + p, v \rangle > k,$$

for all $p \in \partial\Phi(0)$. Similarly as in the proof of (iii) of Theorem 4.6.1, we can find $\sigma > 0$ such that

$$\langle \nabla\mathcal{V}(q(t)) + F(q(t), \dot{q}(t)) + \dot{q}^*(t), v \rangle > k,$$

for all $t \in (t' - \sigma, t' + \sigma), \dot{q}^*(t) \in \partial\Phi(\dot{q}(t))$ due to the continuity of $\mathcal{V} \circ q$ and the graph-closedness property of $\partial\Phi$. It implies that for all $t \in (t' - \sigma, t' + \sigma)$:

$$\langle -M(q(t))\ddot{q}(t) - C(q(t), \dot{q}(t))\dot{q}(t), v \rangle > k.$$

If there exists $0 < \delta < \sigma$ such that $\dot{q}(t) = 0$ for all $t \in (t' - \delta, t' + \delta)$, then $\ddot{q}(t) = 0$ for all $t \in (t' - \delta, t' + \delta)$ and

$$0 = \langle -M(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t), v \rangle > k > 0,$$

for all $t \in (t' - \delta, t' + \delta)$, a contradiction. Hence, t' is isolated in the zero set of $\dot{q}(\cdot)$. ■

Proposition 4.6.3 *Assume that for given initial condition, the system* (4.9) *has a unique solution* $(q(\cdot), \dot{q}(\cdot))$. *Let assumptions of Theorem* 4.6.2 *hold. Then, the set* $D = \{t < t_f : \dot{q}(t) = 0\}$ *is discrete and countable.*

Proof. If $D = \emptyset$, the conclusion is trivial. Otherwise, let $t' \in D$. If $-\mathcal{V}(q(t')) - F(q(t'), 0) \in \partial\Phi(0)$, we must have $q(t) = q(t')$ for all $t \ge t'$, due to the uniqueness of solutions. Hence $t' \ge t_f$, a contradiction with $t' \in D$. Therefore $-\mathcal{V}(q(t')) - F(q(t'), 0) \notin \partial\Phi(0)$. Similarly as in the proof of Proposition 4.6.2, we obtain that t' is isolated in D. Since it is true for each $t' \in D$, we conclude that D is discrete and countable. ■

Example 4.6.1 *Consider the following system in dimension one:*

$$(1 + \cos(q)^2)\ddot{q} - \frac{1}{2}\sin(2q)\dot{q}^2 - g\sin(q) \in -\mathrm{Sign}(\dot{q}), \tag{4.61}$$

where $g = 9.8$ *is the gravity of earth and the set-valued map* $\mathrm{Sign}(\cdot) = \partial|\cdot|$. *In this case,* $M(q) = 1 + \cos(q)^2, C(q, \dot{q}) = -\frac{1}{2}\sin(2q)\dot{q}$ *and* $\mathcal{V}(q) = g\cos(q)$. *It is easy to check the system satisfies the assumptions of Theorem* 4.4.1, *Theorem* 4.5.1 *and Lemma* 4.5.3. *Therefore, for given initial conditions, there exists uniquely a solution satisfying* (4.61) *in the sense of Theorem* 4.3.1 *and the solution converges to the set of equilibria. The following figure shows the numerical simulation of* (4.61) *with* $q(0) = 3, \dot{q}(0) = 1$. *We can see that the trajectory is bounded, hence it converges to a point* q_∞ *belonging the the set of equilibria and* $\dot{q}(\cdot)$ *converges to zero. It can be also checked that* $-\nabla\mathcal{V}(q_\infty) \in \mathrm{Int}(\mathrm{Sign}(0))$, *so the trajectory converges to* q_∞ *in finite time due to Theorem* 4.6.2.

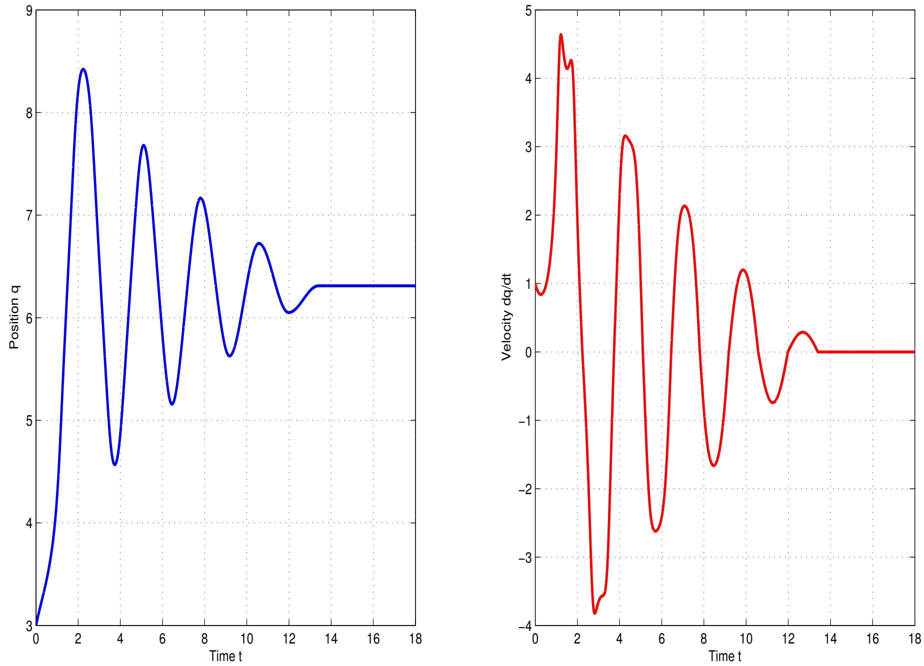


Figure 4.2: Numerical simulation of (4.61) with $q(0) = 3, \dot{q}(0) = 1$.

4.7 Conclusion

In this chapter, the well-posedness of nonlinear Lagrangian dynamical systems with a set-valued controller is analyzed. An existence result is proved by using the Moreau-Yosida regularization. Some conditions ensuring the uniqueness of the trajectory are given. We also study the Lyapunov stability as well as the attractivity properties of the set of stationary solutions of the Lagrangian dynamical systems. We conclude the chapter by giving sufficient conditions ensuring finite-time convergence of the trajectory to an equilibrium point with an estimation of the settling-time. The chapter raises some important questions about the well-posedness, the robustness and the stability analysis of a set-valued controller for the Lagrangian dynamical systems. This problem is known to be difficult and there are only few results in that direction. Our methodology is original and used tools from convex and set-valued analysis. It will be interesting to incorporate the proposed theory in some practical and concrete situation in engineering, and to study the discrete-time version of the discontinuous controllers studied in this chapter and extending these results to infinite dimensional case. It may be a subject of a future work.

Chapter 5

Lur’e Dynamical Systems

5.1 Introduction

Non-monotone set-valued Lur’e dynamical systems have been widely studied in control and applied mathematics (see [81]). The general Lur’e systems are the systems which have a negative feedback interconnection of an ordinary differential equation $\dot{x}(t) = f(x(t), p(t))$ where p is one of the two slack variables, with the second one $q = g(x, p)$ and satisfies the inclusion condition $p \in \Phi(q, t)$. There is a remark that other mathematical models used to study non-smooth dynamical systems (relay systems, evolution variational inequalities, projected dynamical systems, complementarity systems…) can be also recast into Lur’e systems with a set-valued feedback part [34, 52, 65].

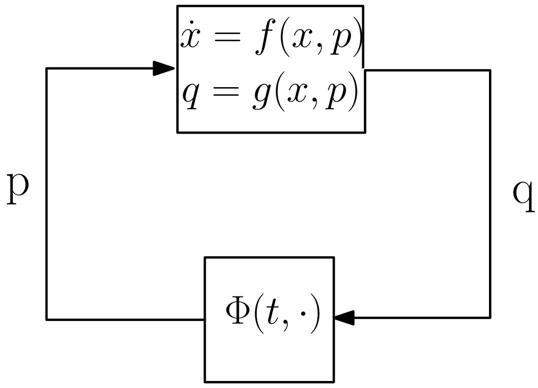


Figure 5.1: Lur’e systems.

We are interested in the Lur’e systems which are (possibly nonlinear) time-invariant dynamical systems with static set-valued feedback. Usually, the function g has the form: $g(x, p) = \mathcal{C}x + Dp$. The case $D = 0$ appears in many applications in electronics particularly, while the case $D \neq 0$ is more general but creates some difficulties when one wants to study the possibly set-valued operator $(-D + \Phi^{-1})^{-1}(\mathcal{C} \circ \cdot)$. In [36], Brogliato and Goeleven overcome these obstacles by assuming that Φ is the sub-differential of some proper, convex, lower semicontinuous function to enjoy the nice properties of Fenchel transform and maximally monotone operators. However, the case of $D = 0$ is still interesting since it includes applications in practice such as the electrical circuits with some devices where their voltage-current characteristics are not monotone. There are certainly many results for the null matrix D, but very few ones for the case of non-monotone set-valued parts [35, 37]. This is the aim of our work, which is hoped to fill some gaps in the study literature of Lur’e systems.

In this chapter, we reformulate a class of Lur'e systems into the first-order differential inclusion form where the set-valued right-hand side is upper semi-continuous with non-empty, convex, compact values to obtain the existence of a solution. Then, local hypo-monotonicity is assumed to ensure the uniqueness result. Next, we give a stability analysis and extend LaSalle's invariance principle to such systems. Finally, some illustrative examples in electronics are presented with numerical simulations. This chapter is a joint work with Prof. S. Adly [14].

5.2 Nonsmooth Lur'e Dynamical System

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a (possibly) nonlinear operator, $B \in \mathbb{R}^{n\times m}, \mathcal{C} \in \mathbb{R}^{m\times n}$ be given matrices; $f \in C(\mathbb{R}^n;\mathbb{R})$ and $\mathcal{F}_i : \mathbb{R} \rightrightarrows \mathbb{R}$ $(i = 1, 2, \dots, m)$ be given upper semi-continuous mappings with non-empty, convex, compact values; $p = (p_1, \dots, p_m)^T, q = (q_1, \dots, q_m)^T :$ $\mathbb{R}^+ \to \mathbb{R}^m$ be two unknown mappings. For $x_0 \in \mathbb{R}^n$, we consider the following problem: Find an absolutely continuous function $x(\cdot)$ defined on $[0, +\infty)$ such that:

$$\begin{cases} x'(t) = A(x(t)) + Bp(t) + f(t) \text{ a.e. } t \in [0, +\infty); \\ \\ q(t) = \mathcal{C}x(t), \\ \\ p_i(t) \in \mathcal{F}_i(q_i(t)), i = 1, 2, \dots, m, \forall\, t \geq 0; \\ \\ x(0) = x_0. \end{cases} \tag{5.1}$$

Particularly in electrical circuits, the inclusion $p_i(t) \in \mathcal{F}_i(q_i(t))$ may represent the voltage-current characteristics of some electronic devices. Note that, $\mathcal{F}_i$ here maybe a non-monotone operator for some $i \in \{1, 2, \dots, m\}$. In practice, we are interested in the state variable $x(\cdot)$. However, in our system, $q(\cdot)$ can be computed uniquely in the term of $x(\cdot)$, so $q(\cdot)$ is also absolutely continuous. The mapping $p(\cdot)$ may be found uniquely by $x(\cdot)$ almost everywhere. Its properties depend on the regularity of the set-valued mapping $\mathcal{F}_i$. For example, it is known that if $\mathcal{F}_i$ is the sub-differential of some proper, convex, lower semicontinuous function for all $i = 1, 2, \dots, m$ then $p(\cdot)$ is absolutely continuous ([36]).

Let $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^m, q = (q_1, \dots, q_m)^T \to F(q)$ defined by:

$$F(q) = \big(\mathcal{F}_1(q_1), .., \mathcal{F}_m(q_m)\big)^T. \tag{5.2}$$

It is easy to see that (5.1) can be rewritten as:

$$\begin{cases} x'(t) = A(x(t)) + Bp(t) + f(t) \text{ a.e. } t \in [0, +\infty); \\ \\ q(t) = \mathcal{C}x(t), \\ \\ p(t) \in F(q(t)), \text{for all t} \geq 0; \\ \\ x(0) = x_0. \end{cases} \tag{5.3}$$

We will show in Lemma 5.2.1 that F is also upper semi-continuous mappings with non-empty, convex and compact values.

Lemma 5.2.1 *If $\mathcal{F}_i : \mathbb{R} \rightrightarrows \mathbb{R}$ is upper semi-continuous mappings with non-empty, convex and compact values for each $i = 1, \ldots, m$ then $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ defined in* (5.2) *is also upper semi-continuous mappings with non-empty, convex and compact values.*

Proof. It is clear that F has non-empty, convex and compact values. It remains to prove that F is upper semicontinuous. Given $\epsilon > 0$ and $x = (x_1, \ldots, x_m)^T$. Then we can choose $\bar{\epsilon} = \epsilon/\sqrt{m} > 0$ such that $\big(\mathbb{B}_{\bar{\epsilon}}(\mathcal{F}_1(x_1)), \ldots, \mathbb{B}_{\bar{\epsilon}}(\mathcal{F}_m(x_m))\big)^T \subset \mathbb{B}_\epsilon\big((\mathcal{F}_1(x_1), \ldots, \mathcal{F}_m(x_m))^T\big) = \mathbb{B}_\epsilon(F(x))$. Since for each $i = 1, \ldots, m$, the mapping $\mathcal{F}_i$ is upper semicontinuous, there exists $\bar{\delta} > 0$ such that $\mathcal{F}_i(\mathbb{B}_{\bar{\delta}}(x_i)) \subset \mathbb{B}_{\bar{\epsilon}}(\mathcal{F}_i(x_i))$. Finally, we take $\delta = \bar{\delta}$ then $\mathbb{B}_\delta(x) \subset (\mathbb{B}_{\bar{\delta}}(x_1), .., \mathbb{B}_{\bar{\delta}}(x_n))^T$and hence $F(\mathbb{B}_\delta(x)) \subset \big(\mathcal{F}_1(\mathbb{B}_{\bar{\delta}}(x_1)), \ldots, \mathcal{F}_m(\mathbb{B}_{\bar{\delta}}(x_m))\big)^T$. So we have found a δ such that $F(\mathbb{B}_\delta(x)) \subset \mathbb{B}_\epsilon(F(x))$. Hence, F is upper semi-continuous. ■

Remark 5.2.1 *1) If $\mathcal{F}_i$ is hypo-monotone (resp. locally hypo-monotone) for each $i = 1, \ldots, m$ then F defined in* (5.2) *is also hypo-monotone (resp. locally hypo-monotone). Indeed for all $x = (x_1, \ldots, x_n)^T, y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$, we have:*

$$\begin{aligned}\langle F(x) - F(y), x - y\rangle &= (\mathcal{F}_1(x_1) - \mathcal{F}_1(y_1))(x_1 - y_1) + \ldots + (\mathcal{F}_n(x_n) - \mathcal{F}_n(y_n))(x_n - y_n) \\ &\geq -k_1|x_1 - y_1|^2 - \ldots - k_n|x_n - y_n|^2 \geq -k\|x - y\|^2.\end{aligned}$$

where $k = \max\{k_1, \ldots, k_n\}$.
2) Particularly, in electrical circuits, many devices have their ampere-volt characteristics of the form: $V \in \partial_C j(i)$, where ∂_C denotes the Clarke's subdifferential and j is a locally Lipschitz function [5]. Note that $\partial_C j$ is then upper semicontinuous with nonempty, convex, compact values [49].
3) In [36], Brogliato and Goeleven consider the systems:

$$\begin{cases} x'(t) = A(x(t)) + Bp(t) + f(t) \text{ a.e. } t \in [0, +\infty); \\ q(t) = \mathcal{C}x(t) + Dp(t), \\ p(t) \in -\partial\Phi(q(t)), \text{for all t} \geq 0; \\ x(0) = x_0, \end{cases} \tag{5.4}$$

where $f \in C(\mathbb{R}_+; \mathbb{R}^n)$ such that $f' \in L^1_{loc}(\mathbb{R}_+; \mathbb{R}^n)$ and $\Phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is given proper convex, lower semicontinuous functions. Then $\partial\Phi$ is maximal monotone and $p(t) \in -\partial\Phi(\mathcal{C}x(t) + Dp(t))$. Define the new function $\Phi^{*,-}(\cdot) = \Phi^*(-\cdot)$, then under certain mild condition, we obtain that $\partial\Phi^{*,-}$ is also maximal monotone and $\partial\Phi^{*,-}(\cdot) = -\partial\Phi^*(-\cdot)$. Hence: $\mathcal{C}x(t) \in -(D + \partial\Phi^{*,-})(p(t))$. Under some conditions on D, we can obtain nice properties of operator $(D + \partial\Phi^{*,-})^{-1}$ and compute $p(t)$ in the term of $x(t)$.

5.3 Existence and Uniqueness

In this section, we recast the Lur'e system into the Filippov case to obtain the existence of a solution. Each solution can be extended globally by using the linear growth condition. Then, local hypo-monotonicity of the right-hand side is supposed to have the uniqueness of the solutions.

Indeed, the system (5.2) can be reduced to the first order differential inclusion:

$$x'(t) \in \mathcal{Q}(t, x(t)) := A(x(t)) + BF(\mathcal{C}x(t)) + f(t) \quad \text{a.e. } t \in [0, +\infty). \tag{5.5}$$

Theorem 5.3.1 *Suppose that A is k-lipschitz and there exists a positive constant c_F such that:*

$$\|w\| \leq c_F(1 + \|y\|), \ \ \forall\, w \in F(y), \ \ \forall y \in \mathbb{R}^m. \tag{5.6}$$

Then for every $x_0 \in \mathbb{R}^n$, there exists a global solution of (5.5)*.*

Proof. It easy to check that A satisfies the linear growth condition. Indeed, we have: $\|A(x)\| \leq \|A(0)\| + k\|x\|$ for all $x \in \mathbb{R}^n$. Note that f is continuous and hence f is locally integrable on $[0, +\infty)$. Then we can imply that, $\mathcal{Q}$ satisfies the conditions of Theorem 2.3.3. Then for all initial condition $x_0 \in \mathbb{R}^n$, there exists an absolutely continuous function $x(\cdot; x_0)$ satisfying:

$$x(0) = x_0,$$

and

$$x'(t) \in \mathcal{Q}(t, x(t)) \quad \text{a.e. } t \in [0, +\infty).$$

Therefore, we obtain the existence of a solution of problem (5.5). ■

Theorem 5.3.2 *If $-F$ is locally hypo-monotone and there exists an invertible matrix $R \in \mathbb{R}^{n\times n}$ such that:*

$$\mathcal{C} = B^T R^T R,$$

then (5.5) *has at most one solution.*

Proof. The system (5.5) is equivalent to:

$$R\dot{x} \in RA(R^{-1}Rx) + RBF(B^T R^T Rx) + Rf(t).$$

Setting $z = Rx$, we get:

$$\dot{z} \in RA(R^{-1}z) + RBF(B^T R^T z) + Rf(t). \tag{5.7}$$

Note that $(RB)^T = B^T R^T$, from Remark 5.2.1 and Lemma 5.2.1 we obtain $-RB \circ F \circ B^T R^T$ is locally hypo-monotone. The mapping $R \circ A \circ R^{-1}$ is Lipschitz. Hence, if for all $z \in \mathbb{R}^n$ we set:

$$\phi(z) = RA(R^{-1}z) + RBF(B^T R^T z),$$

then $-\phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is locally hypo-monotone. Given arbitrary $T > 0$. Suppose that $x_1(\cdot), x_2(\cdot)$ are two solutions of (5.5) on $[0, T]$ with the same initial conditions $x_1(0) =$

$x_2(0) = x_0$. Then $z_i(\cdot) := Rx_i(\cdot), i = 1, 2$ are two solutions of (5.7) satisfying $z_1(0) = z_2(0) = z_0 := Rx_0$. Since $-\phi$ is locally hypo-monotone, there exist $\epsilon > 0$ and $k > 0$ such that $-\phi$ is hypo-monotone with constant k in $\mathbb{B}_\epsilon(z_0)$. Note that, $z_1(\cdot)$ and $z_2(\cdot)$ are absolutely continuous functions. Hence, we can find a positive $T_0 \leq T$ such that $z_1(t), z_2(t) \in \mathbb{B}_\epsilon(z_0)$ for all $t \in [0, T_0]$. From the definition of hypo-monotonicity, we imply that:

$$\langle -\dot{z}_1(t) + \dot{z}_2(t), z_1(t) - z_2(t)\rangle \geq -k\|z_1(t) - z_2(t)\|^2,$$

or, equivalently:

$$\langle \dot{z}_1(t) - \dot{z}_2(t), z_1(t) - z_2(t)\rangle \leq k\|z_1(t) - z_2(t)\|^2,$$

which means:

$$\frac{1}{2}\frac{d}{dt}\|z_1(t) - z_2(t)\|^2 \leq k\|z_1(t) - z_2(t)\|^2.$$

By Gronwall's inequality, we have $\|z_1(t) - z_2(t)\|^2 \leq 0$ for all $t \in [0, T_0]$, *i.e.*, $z_1(t) \equiv z_2(t)$ on $t \in [0, T_0]$. We assume that there exists $t_0 \in [0, T]$ such that $z_1(t_0) \neq z_2(t_0)$. Let:

$$\mathcal{E} = \{t \in [0, t_0] : z_1(t) \neq z_2(t)\}.$$

Since $t_0 \in \mathcal{E}$ and $\mathcal{E}$ is bounded from below, there exists $c = \inf \mathcal{E}$ where $c \in (0, t_0]$ and $z_1(t) = z_2(t)$ for all $t \in [0, c)$. Due to the continuity of $z_1(\cdot), z_2(\cdot)$, we obtain $z_1(c) = z_2(c)$ which means that $c < t_0$. Using the same argument as above, we can find a neighborhood of c on which $z_1(\cdot) \equiv z_2(\cdot)$, a contradiction with the definition of c. Therefore, we have $z_1(\cdot) \equiv z_2(\cdot)$ on $[0, T]$ which implies that $x_1(\cdot) \equiv x_2(\cdot)$ on $[0, T]$. Since $T > 0$ is arbitrary, we have proved the result. ■

5.4 Stability and Invariance Theorems

In the following section, we consider the case of autonomous systems, i.e. $f \equiv 0$. We give some results about the stability of equilibria and a generalized version of Lasalle's invariance principle is presented. Let us denote the set-valued orbital derivative of a continuously differentiable function $V : \bar{\mathbb{B}}_\sigma \subset \mathbb{R}^n \to \mathbb{R}$ (for some $\sigma > 0$) with respect to the differential inclusion (5.5) :

$$\dot{V}(x) = \{p \in \mathbb{R} : \exists \omega \in \mathcal{Q}(x) \text{ such that } p = \langle V'(x), \omega\rangle\}. \tag{5.8}$$

The upper and lower orbital derivatives of V are sequently defined by:

$$\dot{V}^*(x) = \max_{\omega \in \mathcal{Q}(x)} \langle V'(x), \omega\rangle,\ \dot{V}_*(x) = \min_{\omega \in \mathcal{Q}(x)} \langle V'(x), \omega\rangle.$$

Remark 5.4.1 *1) Since $\mathcal{Q}$ has non-empty, convex and compact values, we have $\dot{V}(x)$ is a non-empty, convex compact subset in $\mathbb{R}$. Therefore, $\dot{V}(x)$ is of the following form: $\dot{V}(x) = [\dot{V}_*(x), \dot{V}^*(x)]$. Note that the orbital derivative of more general Lyapunov functions V has been also studied (for example, see [24]).*

2) If $x(t) := x(t; x_0)$ *is a solution of* (5.5) *then:*

$$\frac{d}{dt}V(x(t)) \in \dot{V}(x(t)) \ \ a.e.\ t \geq t_0.$$

3) Let $\bar{x} \in \mathcal{W}$*, i.e.,* $0 \in \mathcal{Q}(\bar{x})$*. From* (5.8)*, it is easy to check that* $0 \in \dot{V}(\bar{x})$*. It means that* $\mathcal{W} \subset \mathcal{Z} = \{y \in \mathbb{R}^n : 0 \in \dot{V}(y)\}$*. This remark may be used in the next section, when we analyze the asymptotic property of the system by using extended LaSalle's invariance principle.*

Definition 5.4.1 *Let* $V : \bar{\mathbb{B}}_\sigma \subset \mathbb{R}^n \to \mathbb{R}$ *be a continuous function such that* $V(0) = 0$*. We say that* V *is positive definite if* $V(x) > 0$ *for all* $x \in \bar{\mathbb{B}}_\sigma \setminus \{0\}$*.*

Definition 5.4.2 *A Lyaponov function for* (5.5) *is a positive definite continuously differentiable function* $V : \bar{\mathbb{B}}_\sigma \subset \mathbb{R}^n \to \mathbb{R}$ *such that* $\dot{V}^*(x) \leq 0$ *for all* $x \in \bar{\mathbb{B}}_\sigma$*.*

Let $\mathcal{S}(x_0)$ be the set of solutions and $\mathcal{W}$ be the set of stationary solutions of (5.5):

$$\mathcal{W} = \{\bar{x} \in \mathbb{R}^n : 0 \in \mathcal{Q}(\bar{x})\}. \tag{5.9}$$

Assumption 5.4.1 $0 \in \mathcal{Q}(0)$*, which means that* $0 \in \mathcal{W}$*.*

Remark 5.4.2 *Let* x^* *be an equilibrium of* (5.5)*, i.e.,* $0 \in \mathcal{Q}(x^*)$*. If we set* $y(\cdot) = x(\cdot) - x^*$*, then the differential inclusion* (5.5) *becomes:*

$$\dot{y}(t) \in \mathcal{Q}_{x^*}(y(t)) := \mathcal{Q}(y(t) + x^*) \quad \text{a.e. } t \in [0, +\infty). \tag{5.10}$$

Note that the function $\mathcal{Q}_{x^*}(\cdot)$ *possesses the same desired properties as* $\mathcal{Q}(\cdot)$ *has. Furthermore,* $0 \in \mathcal{Q}_{x^*}(0)$*, i.e., the trivial solution is an equilibrium of the new differential inclusion* (5.10)*. Therefore, it makes sense to propose Assumption* 5.4.1 *as well as to study the stability properties of the origin.*

Theorem 5.4.1 *Let the assumption of Theorem* 5.3.1 *and Assumption* 5.4.1*. If there exists a Lyapunov function* V *for problem* (5.5)*, then the trivial solution is stable.*

Proof. Since $V : \bar{\mathbb{B}}_\sigma \to \mathbb{R}$ is positive definite continuously differentiable function, there exist a strictly increasing function $\alpha(\cdot) \in C(\mathbb{R}^+; \mathbb{R})$ with $\alpha(0) = 0$ and a positive real number, still denoted by σ such that

$$V(x) \geq \alpha(||x||) \ \text{ for all x} \in \bar{\mathbb{B}}_\sigma.$$

Without loss of generality, let $0 < \varepsilon < \sigma$ and let $c = \alpha(\varepsilon)$. Since V is positive definite, there exists a $\eta > 0$ such that $\mathbb{B}_\eta \subset \Omega_c^\circ = \{x \in \mathbb{R}^n : \ V(x) < c\}$. Let $\delta = \min\{\varepsilon, \eta\}$. Take $x_0 \in \mathbb{B}_\delta$ and $x(t; x_0)$ is a solution of (5.5) satisfying the initial condition $x(0) = x_0$. Suppose that there exists $t_1 \geq 0$ such that $\|x(t_1; x_0)\| \geq \varepsilon$. Since $x(\cdot; x_0)$ is continuous, we may find some t^* satisfying: $\|x(t^*; x_0)\| = \varepsilon$. Then

$$V(x(t^*; x_0)) \geq \alpha(\|x(t^*; x_0)\|) = \alpha(\varepsilon).$$

On the other hand, V is decreasing along the trajectory on the time interval $[0, t^*]$ due to Remark 5.4.1 and the fact that $\dot{V}^*(x) \leq 0$ for all $x \in \bar{\mathbb{B}}_\sigma$. Hence, we have:

$$V(x(t^*; x_0)) \leq V(x_0) < c = \alpha(\varepsilon).$$

Our proof is finished by the contradiction. ■

Theorem 5.4.2 *Let the assumption of Theorem* 5.3.1 *and Assumption* 5.4.1*. If there exists a Lyapunov function V for problem* (5.5) *such that $\dot{V}^*(x) \leq -\lambda V(x)$ for all $x \in \bar{\mathbb{B}}_\sigma$. Then the trivial solution is asymptotic stable.*

Proof. By the Theorem 5.4.1, the trivial solution is stable. Therefore, there exists $\delta > 0$ such that for all $x_0 \in \mathbb{R}^n$ and $\|x_0\| \leq \delta$, we have $x(t; x_0) \in \bar{\mathbb{B}}_\sigma$ for all $t \geq 0$. On the other hand, we have: $\frac{d}{dt}V(x(t)) \in \dot{V}(x(t))$ *a.e.* $t \geq 0$ and $\dot{V}^*(x) \leq -\lambda V(x)$ for all $x \in \bar{\mathbb{B}}_\sigma$. Hence, we have:

$$\frac{d}{dt}V(x(t)) \leq -\lambda V(x(t)),\ a.e.\ t \geq 0.$$

Using simple integration, we obtain:

$$V(x(t)) \leq V(x_0)e^{-\lambda t},\ t \geq 0.$$

Therefore:

$$0 \leq \alpha(\|x(t)\|) \leq V(x_0)e^{-\lambda t},\ t \geq 0.$$

Since $\alpha(\cdot)$ is strictly increasing, we must have:

$$\limsup_{t\to+\infty} \|x(t)\| = 0.$$

Therefore:

$$\lim_{t\to+\infty} \|x(t)\| = 0.$$

■

Remark 5.4.3 *If we have $\dot{V}^*(x) \leq -\lambda V(x)$ for all $x \in \mathbb{R}^n$ then the trivial solution is globally asymptotic stable. The proof is similar to the one in Theorem* 5.4.1.

In the next part of this section, we will generalize the LaSalle's invariance principle to prove the asymptotic stability of the trivial solution. Firstly, we recall some definitions and properties. Let $x_0 \in \mathbb{R}^n$ and $x(t; x_0)$ be a solution of (5.5), denote the *orbit* of x by:

$$\gamma(x) = \{x(t; x_0) :\ t \geq 0\} \subset \mathbb{R}^n,$$

and the *limit set* of x by:

$$\Lambda(x) = \{p \in \mathbb{R}^n :\ \exists\{t_i\}, t_i \to +\infty \text{ as } i \to \infty \text{ and } x(t_i; x_0) \to p\}.$$

A set $S \subset \mathbb{R}^n$ is said *weakly invariant* if and only if for $x_0 \in S$, there exists a solution of (5.5) starting at x_0 contained in S. It is said *invariant* if and only if for $x_0 \in S$, all solutions of (5.5) starting at x_0 are contained in S.

Remark 5.4.4 *(i) If $\gamma(x)$ is bounded, then $\Lambda(x) \neq \emptyset$ and*

$$\lim_{t\to\infty} d(x(t; x_0), \Lambda(x)) = 0.$$

If the right-hand side of (5.5) *is upper semicontinuous with non-empty, convex, compact values, we have a result that the limit set $\Lambda(x)$ is weakly invariant ([9] or [55], p. 129).*

(ii) The set of stationary solutions $\mathcal{W}$ is weakly invariant. Indeed, if $x_0 \in \mathcal{W}$ then the solution $x(t;x_0) = x_0,\ t \geq t_0$, is contained in $\mathcal{W}$. From Remark 5.4.1.3*, $\mathcal{W}$ is a weakly invariant subset of $\mathcal{Z} = \{y \in \mathbb{R}^n : 0 \in \dot{V}(y)\}$.*

(iv) If for each $x_0 \in \mathbb{R}^n$, the set of solutions $\mathcal{S}(x_0)$ has a unique element, then a weakly invariant set is also invariant. Hence, in the following part, we focus on the case of non-unique solutions which is more general.

Theorem 5.4.3 *(Invariance Theorem) Let the assumption of Theorem* 5.3.1*. Suppose that there exists a function $V \in C^1(\mathbb{R}^n;\mathbb{R})$ such that $\dot{V}^* \leq 0$. Let Ω be a compact invariant subset of $\mathbb{R}^n$, $x_0 \in \Omega$ and $x(\cdot;x_0) \in \mathcal{S}(x_0)$ be a solution of* (5.5)*. Let $\mathcal{Z} = \{y \in \mathbb{R}^n : 0 \in \dot{V}(y)\}$ and $\mathcal{M}$ be the largest weakly invariant subset in the closure of $\mathcal{Z}$ then:*

$$\lim_{t\to+\infty} \text{dist}(x(t;x_0), \mathcal{M}) = 0.$$

Proof. Since $x_0 \in \Omega$, and Ω is invariant, we have $\gamma(x) \subset \Omega$. Therefore, $\gamma(x)$ is bounded and:

$$\lim_{t\to+\infty} \text{dist}(x(t;x_0), \Lambda(x)) = 0.$$

It is enough to prove that $\Lambda(x) \subset \bar{\mathcal{Z}}$ due to the weak invariance of $\Lambda(x)$. Note that the function $V(\cdot)$ is C^1, it is bounded on the compact set Ω. From Remark 5.4.1.2, we imply that $V(x(\cdot))$ is decreasing on $\mathbb{R}^+$ since $\dot{V}^* \leq 0$. Therefore, there exists a real number k such that $\lim_{t\to+\infty} V(x(t;x_0)) = k$. For each $p \in \Lambda(x)$, there exist $\{t_i\}, t_i \to \infty$ as $i \to \infty$ and $x(t_i;x_0) \to p$. Then $V(p) = k$ due to the continuity of $V(\cdot)$. Hence $V(p) = k$ for all $p \in \Lambda(x)$. Let $z \in \Lambda(x)$. Since $\Lambda(x)$ is weakly invariant, there exists a solution $\phi(t;z)$ of (5.5) lying in $\Lambda(x)$. Therefore:

$$V(\phi(t;z)) = k,$$

for all $t \geq 0$ which implies:

$$0 = \frac{d}{dt}V(\phi(t;z)) \in \dot{V}(\phi(t;z)),$$

for almost all $t \geq 0$. Hence, we have:

$$\phi(t;z) \in \mathcal{Z},$$

for almost all $t \geq 0$. Since $\phi(\cdot;z)$ is continuous, we obtain:

$$z = \phi(0;z) \in \bar{\mathcal{Z}},$$

and the result follows. ■

Remark 5.4.5 *1) The Theorem is still true if we replace $\mathcal{Z}$ by $\mathcal{Z}_\Omega = \{y \in \Omega : 0 \in \dot{V}(y)\}$. It is enough to check that $\Lambda(x) \subset \bar{\mathcal{Z}}_\Omega$. Indeed, since $\gamma(x) \subset \Omega$ and Ω is compact, we have $\Lambda(x) \subset \Omega$. Therefore, $\Lambda(x) \subset \bar{\mathcal{Z}} \cap \Omega \subset \bar{\mathcal{Z}}_\Omega$.*

2) We can 'ignore' the role of the compact invariant set Ω, provided we know that $x(\cdot;x_0)$ is bounded. Indeed, from the proof above, if we have $x(\cdot;x_0)$ is bounded then $\lim_{t\to+\infty}\text{dist}(x(t;x_0),\Lambda(x)) = 0$ *and we still have $\Lambda(x)\subset\bar{\mathcal{Z}}$. Therefore:*

$$\lim_{t\to+\infty}\text{dist}(x(t;x_0),\mathcal{M}) = 0.$$

3) By using the Invariance Theorem, we can obtain a stronger result than the one in Theorem 5.4.2.

Corollary 5.4.1 *Let the assumption of Theorem* 5.3.1 *and Assumption* 5.4.1.

i) Suppose that there exists a Lyapunov function V for problem (5.5) *such that $\dot{V}^*(x) < 0$ for all $x \in \bar{\mathbb{B}}_\sigma \setminus \{0\}$ and $\dot{V}^*(0) = 0$. Then the trivial solution is asymptotic stable.*

ii) If we have $\dot{V}^*(x) < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, $\dot{V}^*(0) = 0$ and V is radially unbounded then 0 *is globally asymptotic stable.*

Proof. *i*) It is easy to show that $\mathcal{Z} = \{0\}$. We have known that the trivial is stable. Hence, there exists $\delta > 0$ such that for all $x_0 \in \mathbb{R}^n$ and $\|x_0\| \le \delta$, we have $x(t;x_0) \in \bar{\mathbb{B}}_\sigma$ for all $t \ge 0$, *i.e.*, $x(\cdot;x_0)$ is bounded. The asymptotical stability of the trivial solution follows by using the Remark 5.4.5.

ii) Given $x_0 \in \mathbb{R}^n$, set $\Omega := \{x \in \mathbb{R}^n : V(x) \le V(x_0)\}$ then $x_0 \in \Omega$ and Ω is a compact subset of $\mathbb{R}^n$ since V is radially unbounded. It is sufficient to prove that it is invariant with respect to (5.5). Indeed, let $z \in \Omega$ and $x(\cdot;z)$ be a solution of (5.5) satisfying $x(0;z) = z$. Since the mapping $V(x(t;z))$ is decreasing with respect to t, we have $V(x(t;z)) \le V(x(0;z)) = V(z) \le V(x_0)$ for all $t \ge 0$. Hence $x(t;z) \in \Omega$ for all $t \ge 0$. The result follows by the Invariance Theorem with $\mathcal{Z} = \{0\}$. ■

Corollary 5.4.2 *Let the assumption of Theorem* 5.3.1. *Suppose that there exists a radially unbounded function $V \in C^1(\mathbb{R}^n;\mathbb{R})$ such that $\dot{V}^* \le 0$. Let $\mathcal{Z} = \{y \in \mathbb{R}^n : 0 \in \dot{V}(y)\}$ and $\mathcal{M}$ be the largest weakly invariant subset in the closure of $\mathcal{Z}$. Then for any $x_0 \in \mathbb{R}^n$ and $x(\cdot;x_0) \in \mathcal{S}(x_0)$ a solution of* (5.5), *we have:*

$$\lim_{t\to+\infty}\text{dist}(x(t;x_0),\mathcal{M}) = 0.$$

Proof. For each $x_0 \in \mathbb{R}^n$, set $\Omega := \{x \in \mathbb{R}^n : V(x) \le V(x_0)\}$ then $x_0 \in \Omega$ and Ω is an invariant compact subset of $\mathbb{R}^n$(see the proof of Proposition 5.4.1.*ii*). Hence, the conclusion follows by the Invariance Theorem. ■

Theorem 5.4.4 *Consider the differential* (5.5) *with $A \in \mathbb{R}^{n\times n}$. Suppose that F is upper semi-continuous with non-empty convex, compact values satisfying the linear growth condition and $-F$ is locally hypo-monotone. Furthermore:*

(i) there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a matrix $K \in \mathbb{R}^{n\times l}$ for some integer $l > 0$ such that: $PA + A^TP = -KK^T$ and $\mathcal{C}^T = PB$;

(ii) $\sup_{w\in F(x)}\langle x, w\rangle \leq 0$ *for all* $x \in \mathbb{R}^n$.

Then for each $x_0 \in \mathbb{R}^n$, *there exists a unique solution* $x(t; x_0)$ *of* (5.5). *Furthermore, let* $V(y) := \frac{1}{2}y^T Py$, $y \in \mathbb{R}^n$; $\mathcal{Z} = \{y \in \mathbb{R}^n : 0 \in \dot{V}(y)\}$ *and* $\mathcal{M}$ *be the largest invariant subset of* $\bar{\mathcal{Z}}$. *Then:*

$$\lim_{t\to+\infty} \operatorname{dist}(x(t; x_0), \mathcal{M}) = 0.$$

Proof. It is obvious that all the assumptions of Theorem 5.3.1 and 5.3.2 are satisfied. Indeed, we can choose the matrix $R = \sqrt{P}$. Therefore, for each $x_0 \in \mathbb{R}^n$, there exists a unique solution $x(t; x_0)$ of (5.5). Note that V is radially unbounded and its upper orbital derivative is:

$$\dot{V}^*(x) = \sup_{w\in F(\mathcal{C}x)}\langle Px, Ax+Bw\rangle = \langle x, PAx\rangle + \sup_{w\in F(\mathcal{C}x)}\langle B^T Px, w\rangle = \langle x, PAx\rangle + \sup_{w\in F(\mathcal{C}x)}\langle \mathcal{C}x, w\rangle.$$

From (ii), we have $\sup_{w\in F(\mathcal{C}x)}\langle \mathcal{C}x, w\rangle$ is non-positive. On the other hand, the term $\langle x, PAx\rangle = \frac{1}{2}\langle x, (PA+A^Tp)x\rangle = -\frac{1}{2}\langle x, KK^Tx\rangle = -\frac{1}{2}\langle K^Tx, K^Tx\rangle \leq 0$. It implies that the upper orbital derivative of V is non-positive. The conclusion follows by Corollary 5.4.2. ■

Remark 5.4.6 *1) If* $l = m$, *we can use the Kalman-Yakubovich-Popov Lemma* ([5, 38]) *to obtain the existence of* P *and* K *satisfying the condition* (i) *in Theorem* 5.4.4.

2) The condition (ii) *means* x *is contained in the polar cone of* $F(x)$ *for all* $x \in \mathbb{R}^n$. *The geometrical meaning in scalar case is that the graph of* $-F$ *belongs to the first and the third quadrants. It holds for a large class of function, for examples, when* $-F$ *is the characteristic function of a diode, Zener diode, diac, silicon controller rectifier* ([5])*...*

3) If L^T *is full column rank then* LL^T *is positive definite. Then it is easy to check that* $\mathcal{Z} = \{0\}$. *Therefore:*

$$\lim_{t\to+\infty} x(t; x_0) = 0.$$

4) If $\sup_{x\in\mathbb{R}^n, w\in\mathcal{F}_i(x)}\langle x, w\rangle \leq 0$ *for each* $i = 1, 2, \ldots, m$ *then* F *defined in* (5.2) *satisfies* (ii).

Corollary 5.4.3 *Consider the differential* (5.1) *with* $A \in \mathbb{R}^{n\times n}$ *and* $\mathcal{F}_i$ *is upper semi-continuous with non-empty convex, compact values satisfying the linear growth condition and* $-\mathcal{F}_i$ *is locally hypo-monotone for* $i = 1, 2, \ldots, m$. *Suppose that:*

(i) there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ *and a matrix* $K \in \mathbb{R}^{n\times l}$ *for some integer* $l > 0$ *such that:* $PA + A^TP = -KK^T$ *and* $\mathcal{C}^T = PB$;

(ii) $\sup_{w\in\mathcal{F}_i(y)}\langle y, w\rangle \leq 0$ *for all* $y \in \mathbb{R}$, $i = 1, 2, \ldots, m$.

Then for each $x_0 \in \mathbb{R}^n$, *there exists a unique solution* $x(t; x_0)$ *of* (5.1). *Furthermore, let* $V(y) := \frac{1}{2}y^TPy$, $y \in \mathbb{R}^n$; $\mathcal{Z} = \{y \in \mathbb{R}^n : 0 \in \dot{V}(y)\}$ *and* $\mathcal{M}$ *be the largest invariant subset of* $\bar{\mathcal{Z}}$. *Then:*

$$\lim_{t\to+\infty} \operatorname{dist}(x(t; x_0), \mathcal{M}) = 0.$$

Proof. Let $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ defined in (5.2) and using Theorem 5.4.4. ■

5.5 Some Examples in Electronics

Electrical devices can be illustrated by their corresponding ampere-volt characteristics. Each device may possess various mathematical models based on the experimental measures. Figures 3.5 and 3.6 present the characteristics of a diac and a silicon controller rectifier which will be used in some circuit examples later (for reference about the ampere-volt characteristics of electrical devices, see [5]). In both cases, the mappings (f_{diac} and f_{scr}) are set-valued where the values at 0 are intervals in $\mathbb{R}$ and single-valued differentiable with locally bounded derivatives in $\mathbb{R} \setminus \{0\}$. Hence, it can be checked that f_{diac} and f_{scr} are locally hypo-monotone. Furthermore, they are upper semicontinuous (their graphs are closed and locally bounded) with non-empty convex, compact values. They also have a property that for all $x \in \mathbb{R}$, the mappings $xf_{diac}(-x)$ and $xf_{scr}(-x)$ are single-valued and non-positive.

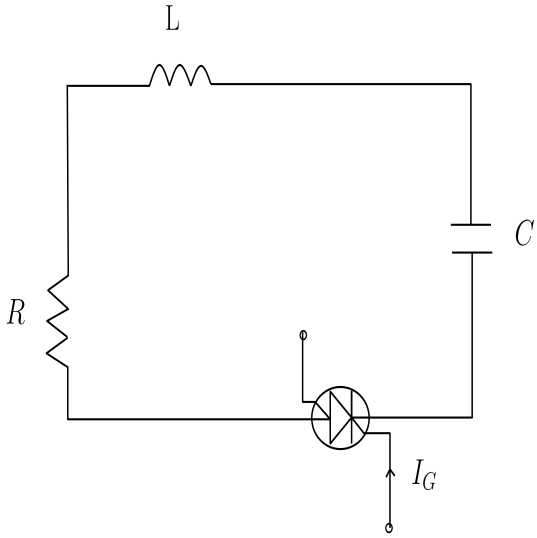


Figure 5.2: Circuit with silicon controller rectifier.

Example 5.5.1 *Consider the circuit in figure* 5.2 *with a resistor* $R > 0$*, an inductor* $L > 0$ *, a capacitor* $C > 0$*. Let* x_1 *be the time integral of the current across the capacitance,* x_2 *the current across the circuit,* y_L *is the voltage of the silicon controlled rectifier. Using Kirchhoff's circuit laws, we have:*

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \overbrace{\begin{pmatrix} 0 \\ -\frac{1}{L} \end{pmatrix}}^{B} y_L, \tag{5.11}$$

where

$$y = \overbrace{(0 \quad -1)}^{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad y_L \in f_{scr}(y) = f_{scr}(-x_2). \tag{5.12}$$

Let $\mathcal{Q} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2, \mathcal{Q}(x) = Ax - Bf_{scr}(\mathcal{C}x)$ *and* $F := -f_{scr}$*. The mapping* $\mathcal{Q}$ *is upper semi-continuous with non-empty convex compact values and the mapping* $-F = f_{scr}$ *is locally hypo-monotone. The matrix* R *defined by:*

$$R = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{L} \end{pmatrix}$$

holds $\mathcal{C} = B^T R^T R$*. Therefore, the assumptions of Theorem* 5.3.1, 5.3.2 *and* 5.4.1 *are satisfied. Let* $V(x) = \frac{\alpha_1}{2}x_1^2 + \frac{\alpha_2}{2}x_2^2$ *where* $\alpha_1, \alpha_2 > 0$ *can be chosen later, then* V *is* C^1*, radially unbounded and:*

$$V'(x) = \begin{pmatrix} \alpha_1 x_1 \\ \alpha_2 x_2 \end{pmatrix}.$$

We have:

$$\dot{V}(x) = \alpha_1 x_1 x_2 + \alpha_2 x_2(-\frac{1}{LC}x_1 - \frac{R}{L}x_2 + \frac{1}{L}f_{scr}(-x_2))$$

is single-valued since $x_2 f_{scr}(-x_2)$ *is single-valued. We choose* $\alpha_1 = \frac{\alpha_2}{LC}$*, then we obtain that:*

$$\dot{V}^*(x) = \dot{V}_*(x) = -\frac{R\alpha_2}{L}x_2^2 + \frac{\alpha_2}{L}x_2 f_{scr}(-x_2) \leq 0.$$

It is easy to check that $\mathcal{Z} = \{y \in \mathbb{R}^2 : 0 \in \dot{V}(y)\} = \mathbb{R} \times \{0\}$ *and the set of stationary solutions of the system* $\mathcal{W} = \{(x_1, 0) : x_1 \in \mathbb{R},\ x_1 \in Cf_{scr}(0)\} = Cf_{scr}(0) \times \{0\}$*. We can prove that* $\mathcal{W}$ *is the largest invariance subset of* $\mathcal{Z}$*. Indeed, let* $\mathcal{D}$ *be an invariance subset of* $\mathcal{Z}$ *and* $z = (z_1\ z_2)^T \in \mathcal{D}$*. The unique solution* $\psi(\cdot; z) = (\psi_1(\cdot; z)\ \psi_2(\cdot; z))^T$ *of* (5.11) *satisfies* $\psi(t; z) \in \mathcal{D}$ *for all* $t \geq 0$*. Then for all* $t \geq 0$*, we have* $\dot{\psi}_1(t; z) = \psi_2(t; z) = 0$ *which implies* $\psi_1(t; z) \equiv z_1, z_2 = 0$ *and* $z_1 \in Cf_{scr}(0)$*. Therefore, we have:* $\mathcal{D} \subset \mathcal{Z}$*. Using Proposition* 5.4.2*, we obtain:*

$$\lim_{t\to+\infty} \text{dist}(x_1(t), Cf_{scr}(0)) = 0 \text{ and } \lim_{t\to+\infty} x_2(t) = 0.$$

Consider the circuit above which includes a voltage supply u*. Then, the system becomes:*

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ -\frac{1}{L} \end{pmatrix} y_L + \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix} u, \tag{5.13}$$

and

$$\dot{V}^*(x) = \frac{d}{dt}V(x) = -\frac{R\alpha_2}{L}x_2^2 + \frac{\alpha_2}{L}x_2 y_L + \frac{\alpha_2 u}{L}x_2.$$

Let $c := \inf_{x\neq 0} |f_{scr}(x)| > 0$*. If* $|u| \leq c$*, we have* $\frac{\alpha_2}{L}x_2 y_L + \frac{\alpha_2 u}{L}x_2 \leq \frac{\alpha_2}{L}|x_2|(|u| - |y_L|) \leq 0$*. Therefore,* $\dot{V}^*(x) \leq 0$ *and we obtain the stability of the trivial solution and the attractivity result as above. The set of stationary solutions then is* $\mathcal{W} = \{(x_1, 0) : x_1 \in \mathbb{R},\ x_1/C \in f_{scr}(0) + u\} = C(f_{scr}(0) + u) \times \{0\}$ *and we have:*

$$\lim_{t\to+\infty} \text{dist}(x_1(t), C(f_{scr}(0) + u)) = 0 \text{ and } \lim_{t\to+\infty} x_2(t) = 0.$$

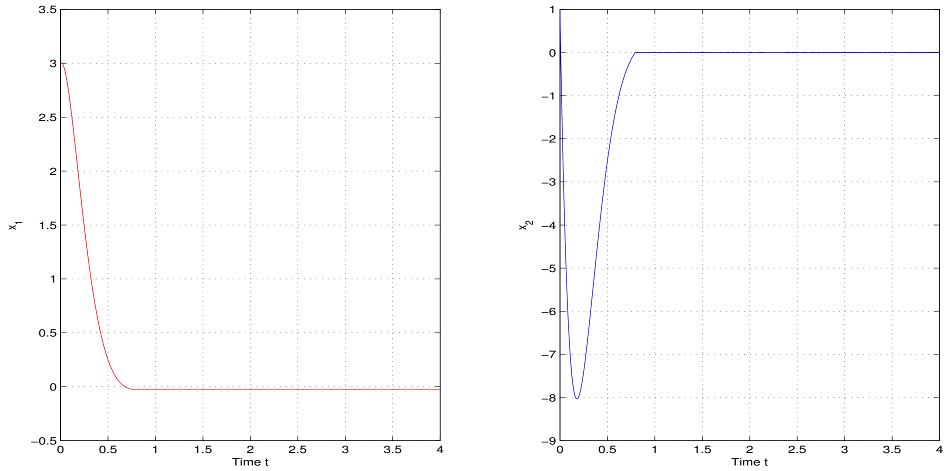


Figure 5.3: Numerical simulation for the circuit described by figure 5.2.

Note that we can use directly Theorem 5.4.4 *with:*

$$P = \begin{pmatrix} 1/C & 0 \\ 0 & L \end{pmatrix} \text{ and } K = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2R} \end{pmatrix},$$

which satisfy $PA + A^T P = -KK^T$ *and* $\mathcal{C}^T = PB$. *The same analysis can be applied if we replace the function* f_{scr} *by* f_{diac}.

Example 5.5.2 *Next, we consider the circuit correspondent to the figure* 5.4. *Applying Kirchhoff's circuit laws again, we obtain:*

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{L_2 C} & -\frac{R_1+R_3}{L_2} & \frac{R_1}{L_2} \\ 0 & \frac{R_1}{L_1} & -\frac{R_1+R_2}{L_1} \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \overbrace{\begin{pmatrix} 0 & 0 \\ \frac{1}{L_2} & \frac{1}{L_2} \\ -\frac{1}{L_1} & 0 \end{pmatrix}}^{B} \begin{pmatrix} y_{L_1} \\ y_{L_2} \end{pmatrix} \tag{5.14}$$

and

$$y_{L_1} \in f_{diac}(-x_3 + x_2) \text{ and } y_{L_2} \in f_{scr}(x_2),$$

where $R_1, R_2, R_3 > 0$ *are resistors,* $L_1, L_2 > 0$ *are inductors,* $C > 0$ *is a capacitor,* x_1 *is the time integral of the current across the capacitor,* x_2 *is the current across the capacitor,* x_3 *is the current across the inductor* L_1, y_{L_1} *is the voltage of the diac,* y_{L_2} *is the voltage of the SCR. Let* $\mathcal{C} := \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ *and* $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2, x = (x_1\ x_2)^T \to -\begin{pmatrix} f_{diac}(x_1) \\ f_{scr}(x_2) \end{pmatrix}$. *Denote* $\mathcal{Q} : \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$ *the right-hand side of* (5.14), *then for* $x = (x_1\ x_2\ x_3)^T$ *we have:*

$$\mathcal{Q}(x) = Ax + BF(\mathcal{C}x).$$

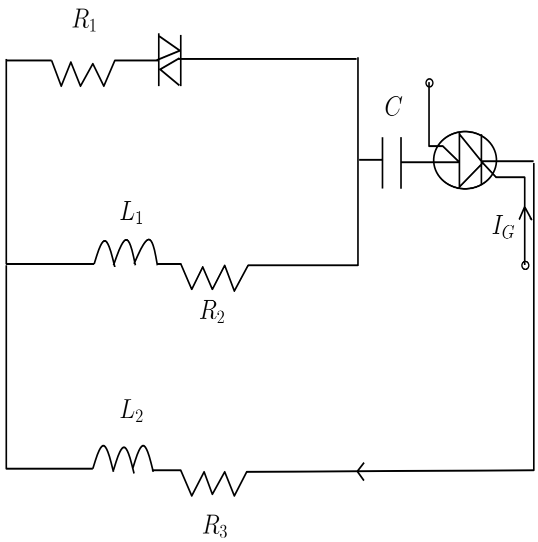


Figure 5.4: Circuit with silicon controller rectifier and diac.

It is clear that $\mathcal{Q}$ is upper semi-continuous with non-empty convex compact values and the mapping $-F$ is locally hypo-monotone. The following matrix:

$$R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{L_2} & 0 \\ 0 & 0 & \sqrt{L_1} \end{pmatrix}$$

satisfies $\mathcal{C} = B^T R^T R$. Hence, the assumptions of theorems 5.3.1, 5.3.2 and 5.4.1 hold. We also consider the Lyapunov function of the form:

$$V(x) = \frac{\alpha_1}{2}x_1^2 + \frac{\alpha_2}{2}x_2^2 + \frac{\alpha_3}{2}x_3^2,$$

where $\alpha_1, \alpha_2, \alpha_3 > 0$ can be chosen later. Let

$$\beta_1 = \frac{1}{L_2C}, \beta_2 = \frac{R_1+R_2}{L_2}, \beta_3 = \frac{R_1}{L_2}; \beta_4 = \frac{1}{L_2}, \beta_5 = \frac{R_1}{L_1}, \beta_6 = \frac{R_1+R_2}{L_1}, \beta_7 = \frac{1}{L_1}.$$

Then

$$\begin{aligned} \dot{V}(x) &= \alpha_1 x_1 x_2 + \alpha_2 x_2(-\beta_1 x_1 - \beta_2 x_2 + \beta_3 x_3 - \beta_4 f_{diac}(-x_3+x_2) - \beta_4 f_{scr}(x_2)) \\ &+ \alpha_3 x_3(\beta_5 x_2 - \beta_6 x_3 + \beta_7 f_{diac}(-x_3+x_2)). \end{aligned} \tag{5.15}$$

We choose $\alpha_1, \alpha_2, \alpha_3$ such that $\alpha_1 = \alpha_2\beta_1, \alpha_2\beta_4 = \alpha_3\beta_7 := \gamma$ then we obtain:

$$\dot{V}(x) = (\alpha_2\beta_3 + \alpha_3\beta_5)x_2x_3 - \alpha_2\beta_2x_2^2 - \alpha_3\beta_6x_3^2 + \gamma(x_3 - x_2)f_{diac}(-x_3+x_2) - \alpha_2\beta_4x_2f_{scr}(x_2) \tag{5.16}$$

is single valued since $(x_3 - x_2)f_{diac}(-x_3+x_2)$ and $x_2f_{scr}(x_2)$ are single valued.

Replacing $\alpha_3 = \alpha_2\frac{\beta_4}{\beta_7}$, we have that:

$$\alpha_2\beta_3+\alpha_3\beta_5 = (\beta_3+\frac{\beta_4\beta_5}{\beta_7})\alpha_2 = \frac{2R_1}{L_2}\alpha_2, \beta_2 = \frac{R_1+R_3}{L_2} > \frac{R_1}{L_2}, \alpha_3\beta_6 = \frac{\beta_4\beta_6}{\beta_7}\alpha_2 = \frac{R_1+R_2}{L_2}\alpha_2 > \frac{R_1}{L_2}\alpha_2.$$

Therefore:

$$(\alpha_2\beta_3+\alpha_3\beta_5)x_2x_3-\alpha_2\beta_2x_2^2-\alpha_3\beta_6x_3^2\leq\frac{R_1}{L_2}\alpha_2(2x_2x_3-x_2^2-x_3^2)=-\frac{R_1}{L_2}\alpha_2(x_2-x_3)^2\leq 0.$$

Note that:

$$(x_3-x_2)f_{diac}(-x_3+x_2)\leq 0 \text{ and } -x_2f_{scr}(x_2)\leq 0.$$

So we have:

$$\dot{V}^*(x)=\dot{V}_*(x)\leq 0.$$

We can check that $\mathcal{Z}=\{y\in\mathbb{R}^3:0\in\dot{V}(y)\}=\mathbb{R}\times\{0\}\times\{0\}$. *The set of stationary solutions* $\mathcal{W}$ *is a subset of* $\mathcal{Z}$ *and* $\mathcal{W}=\{(x_1,0,0):x_1\in\mathbb{R},\ x_1\in -C(f_{diac}(0)+f_{scr}(0)\}=-C(f_{diac}(0)+f_{scr}(0))\times\{0\}\times\{0\}$. *Similarly, we can prove that* $\mathcal{W}$ *is the largest invariant subset of* $\mathcal{Z}$ *and by using Proposition* 5.4.2, *we have:*

$$\lim_{t\to+\infty}\text{dist}(x_1(t),-C\{f_{diac}(0)+f_{scr}(0)\})=0,\ \lim_{t\to+\infty}x_2(t)=0 \text{ and } \lim_{t\to+\infty}x_3(t)=0.$$

We may also apply Theorem 5.4.4 *for this circuit, which is left as an exercise for readers.*

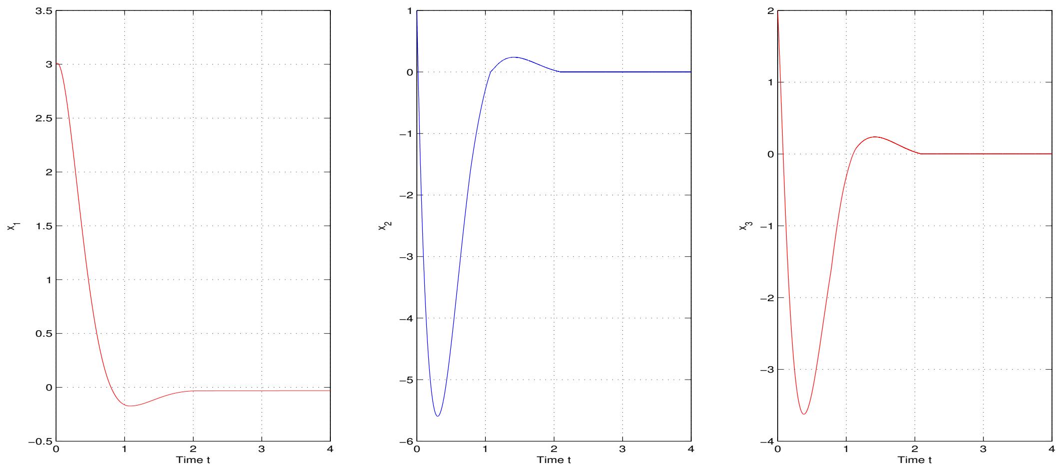


Figure 5.5: Numerical simulation for the circuit described by figure 5.4.

5.6 Conclusion

In this chapter, we have analyzed the well-posedness of a class of non-motonone set-valued Lur'e dynamiacal systems. The existence and uniqueness of the trajectories are assured with a weaker assumption than the case of maximal monotone right-hand side. Some criterions for the Lyapunov function is also given to obtain the stability analysis and asymptotic properties of such systems. We developed an extended version of LaSalle's invariance principle which can be applied to study the attractivity of a stationary solution or a set.

CHAPTER 6

Conclusions and Perspectives

6.1 Conclusions

It is undoubted that non-smooth phenomena play an indisputably important role in many concrete problems in various fields such as mechanics, engineering sciences, electrical circuits, biology... The non-smoothness is intrinsic and it may originate from the dry friction, the switching devices, the environment (non-smooth impact), or from the discontinuous control terms... Classical mathematical methods are no longer useful tools to correctly study these problems. Therefore, it is necessary to develop new analytical tools, say non-smooth analysis, to better capture the qualitative behavior of the real physical systems of interest. Indeed, the development of non-smooth analysis has been explosive in recent decades, allow scientists to consider complicated problems.

We have explained why it is interesting and necessary to study stability analysis of non-smooth dynamical systems. We have also presented in this monograph the good frameworks to study some important classes of dynamical systems in electrical circuits and mechanics with dry frictions. The analysis for such systems are made thoroughly with the support of numerical simulations. They are about the well-posedness of the mathematical models, stability properties, even finite-time stability of trajectories. However, the uniqueness of solutions remains open for a large class of NSDS models or it holds under very restrictive conditions. The lack of uniqueness may be due to the ineffectiveness of the framework, the lack of powerful tools or not enough data to determine the trajectories. It is worthy to note that the method used here can be applied to analyze for non-smooth dynamical systems from other fields.

6.2 Perspectives

Although the results presented in this monograph are new, some of them are original but they are due to the properties of the systems and the Lyapunov functions used herein are smooth. Hence, there is a possibility to apply non-smooth Lyapunov functions for larger class of dynamical systems. Furthermore, we may also use mathematical framework of measure differential inclusions to analyze the systems which have jumps in the states. Compare to the modern stability theory with a great and consistent development, it is certain that this monograph is still very modest. Therefore, there are many modern stability topics such as input-output stability, input-to-state stability, absolute stability, convergent dynamics... to study which may be considered as a continuation of this work. In addition, it is very interesting if we can use the mathematical frameworks discussed in this monograph to study the dynamical systems in other domains such as economics, biology... They all worth to be the subjects of a work in future.

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Notation

$\mathbb{N}$	the natural numbers
$\mathbb{R}$	the real numbers
$\bar{\mathbb{R}}$	the extended real numbers
$\Vert \cdot \Vert$	Euclidean norm
$\Vert \cdot \Vert_1$	1-norm
$\Vert \cdot \Vert_m$	the induced matrix norm
$\mathbb{B}$	closed unit ball
$\mathbb{B}_\varepsilon$	closed ball with radius ε
$\mathbb{B}_\varepsilon(x)$	closed ball with radius ε centered at x
$\langle x, y\rangle$	canonical inner product
$\mathcal{C}^k$	k-continuously differentiable
L^p	L^p space
$\mathcal{W}^{k,p}$	Sobolev space
$A \setminus B$	relative complement
$\mathrm{bd}(A)$	boundary of the set A
$\mathrm{cl}(A)$	closure of the set A
$\mathrm{int}(A)$	interior of the set A
$\mathrm{d}(x, A)$	distance from the point x to the set A
$\mathrm{co}(A)$	convex hull of the set A
$\overline{\mathrm{co}}(S)$	closed convex hull of the set A
$\mathrm{cone}(S)$	conic hull of the set A
$\mathrm{dom}(f)$	effective domain of the function f
$\mathrm{epi}(f)$	epigraph of the function f
$\mathrm{Graph}(f)$	graph of the function f
$f^0(x; v)$	the generalized directional derivative of f at x
$T_C(x)$	tangent cone at x of the set C
$N_C(x)$	normal cone at x of the set C
K^*	dual cone of the set K
K^0	polar cone of the set K
f^*, f^{**}	conjugate, biconjugate of the function f
δ_C	indicator function of of the set C
σ_C	support function of of the set C

Résumé : L'objectif principal de cette thèse est de proposer une formulation pour l'étude et l'analyse de stabilité des systèmes dynamiques non-réguliers avec une attention particulière aux applications issues des circuits électriques et des systèmes mécaniques avec frottement sec. Les outils mathématiques utilisés sont issus de l'analyse non-lisse et de la théorie de stabilité au sens de Lyapounov. Dans le détail, nous utilisons un formalisme pour modéliser la complémentarité des systèmes de commutation simples et des inclusions différentielles pour modéliser un convertisseur DC-DC de type Buck, les systèmes dynamiques Lagrangian ainsi que les systèmes de Lur'e. Pour chaque modèle, nous nous intéressons à l'existence d'une solution, des propriétés de stabilité des trajectoires, de la stabilité en temps fini ou de mettre une force sur la commande pour obtenir la stabilité en temps fini. Nous proposons aussi quelques méthodes numériques pour simuler ces systèmes. Il est à noter que les méthodes utilisées dans ce manuscrit peuvent être appliquées pour l'analyse de systèmes dynamiques non-réguliers issus d'autres domaines tels que l'économie, la finance ou la biologie ...

Mots-clés : Analyse convexe, analyse variationnelle et multivoque, la stabilité de Lyapunov, convertisseur DC-DC de type Buck , systèmes dynamiques Lagrangiens, les systèmes Lur'e, les systèmes de complémentarité, inclusions différentielles.

Abstract: This manuscript deals with the stability of non-smooth dynamical systems and applications. More precisely, we aim to provide a formulation to study the stability analysis of non-smooth dynamical systems, particularly in electrical circuits and mechanics with dry friction and robustness. The efficient tools which we have used are non-smooth analysis, Lyapunov stability theorem and non-smooth mathematical frameworks: complementarity and differential inclusions. In details, we use complementarity formalism to model some simple switch systems and differential inclusions to model a Dc-Dc Buck converter, Lagrange dynamical systems and Lur'e systems. For each model, we are interested in the well-posedness, stability properties of trajectories, even finite-time stability or putting a control force to obtain finite-time stability, and finding numerical ways to simulate the systems. The theoretical results are supported by some examples in electrical circuits and mechanics with numerical simulations. It is noted that the method used in this monograph can be applied to analyze for non-smooth dynamical systems from other fields such as economics, finance or biology...

Keywords: Convex analysis, set-valued and variational analysis, Lyapunov stability, Dc-Dc Buck converter, Lagrange dynamical systems, Lur'e systems, complementarity systems, differential inclusions.