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Confirmation

I confirm that all the results of this thesis come from my work under the supervision of Professors Phan Quoc Khanh and Samir Adly. They have never been published by other authors.

Hochiminh City, August 2011

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Foreword

The celebrated Ekeland's variational principle (Ekeland 1974) (EVP, from now on) is one of the most important results and cornerstones of nonlinear analysis with applications in many fields of analysis, optimization and operations research. Its importance is emphasized by the fact that there are a number of equivalent formulations, all of which are well known with significant applications and many of which were discovered independently, namely the Caristi-Kirk fixed-point theorem (Caristi 1976), the drop theorem of Daneš (Daneš 1972), the Takahashi theorem about the existence of minima (Takahashi 1991), the petal theorem of Penot (Penot 1986), the Krasnoselski-Zabrejko theorem on solvability of operator equations (Zabreiko and Krasnoselski 1971), Phelps' lemma (Phelps 1974), etc.

Over more than three decades a good deal of effort has been made to look for equivalent formulations or generalizations of the EVP.

The seminal EVP (Ekeland 1974) says roughly that, for a lower semicontinuous (lsc) and bounded from below function f on a complete metric space X, a slightly perturbed function has a strict minimum. Moreover, if X is a Banach space and f is Gateaux differentiable, then its derivative can be made arbitrarily small.

We can first observe generalizations of the EVP to vector minimization, i.e. to the case where f is a mapping with a multidimensional range space Y, see e.g. Loridan (1984), Valyi (1985), Khanh (1989). Here Y may be even an ordered vector space. Extensions of X to the case of topological vector spaces, uniform spaces or L-spaces are investigated e.g. in Khanh (1989), Hamel (2003, 2005), Hamel and Löhne (2006), Qui (2005). In this research direction, a general partial order is often proposed and a minimal point with respect to (wrt) this order is proved to exist, leading to a type of the EVP, see also Göpfer et al. (2000). Smooth variants of the EVP are studied e.g. in Borwein and Preiss (1987), Li and Shi (2000). The second conclusion of Ekeland in the seminal work Ekeland (1974), that (when X is a normed space and f is Gateaux differentiable) f' can be made arbitrarily small, has been attracted also much attentions, see e.g. Ha (2003, 2005, 2006), Bao and Mordukhovich (2007). Here various kinds of generalized derivatives are taken into account: the Fre'chet, Clarke and Mordukhovich coderivatives; the Fre'chet, Clarke and Mordukhovich subdifferentials. Fre'chet Hessians are also used to establish the Ekeland principle for second-order optimality conditions (Arutyunov et al. 1997). Stability results for the EVP are obtained e.g. in Attouch and Riahi (1993), Huang (2001, 2002). In connection with the EVP, existence conditions for optimal solutions in problems with noncompact feasible sets are dealt with in Ha (2003, 2006), Bao and Mordukhovich (2007), El Amrouss (2006) using generalizations of coercivity assumptions, the Palais-Smale condition or the Cerami condition. One of the recent research interests is to consider the case where X is a metric space but equipped with an additional generalized distance, based on which the semicontinuity assumption of Ekeland can be weakened. w-distance was introduced in Kada et al. (1996) and used also in Park (2000), Lin and Du (2007). In Tataru (1992) another distance was proposed to obtain a generalization of the EVP. In Suzuki (2001, 2005) τ-distance, which is more general than both afore-mentioned distances, was introduced to improve the EVP. τ-function was proposed and employed in Lin and Du (2006). The purpose of this thesis is to investigate the topic in both theory and application aspects.

The thesis consists of five chapters. In Chapter 1 we propose a definition of lower closed transitive relations and prove the existence of minimal elements for such a relation. This result is shown to contain probably a large part of existing versions of Ekeland's variational principle (EVP). We introduce the notion of a weak τ-function p as a generalized distance and use it together with the above result on minimal elements to establish enhanced EVP for various settings, under relaxed lower semicontinuity assumptions. These principles conclude the existence not only of p-strict minimizers of p-perturbations of the considered vector function, but also p-sharp and p-strong minimizers. We include equivalent formulations of our enhanced EVP as well. In Chapter 2, using weak τ-functions we investigate the EVP for Kuroiwa minimizers of a set-valued mapping (it is not understood in the usual Pareto sense, but in a meaning recently employed in Kuroiwa (2001), Ha (2005), Hamel (2006)) and equivalent results. In Chapter 3, we propose relaxed lower semicontinuity properties for set-valued mappings, using weak τ-functions, and employ them to weaken known lower semicontinuity assumptions to get enhanced Ekelands variational principle for Pareto minimizers of set-valued mappings and underlying minimal-element principles. For a brief of Chapter 4, first note that, dealing with the EVP for a vector function $f : X \to Y$, there have been three approaches. The first one is applying scalarization techniques to convert the vector problem to a scalar case. The second method is introducing a vector metric $d : X \times X \to Y$ to write versions of vector EVP similar to scalar ones. The third way, which has recently most been applied, including also in our first three chapters, is using some $k_0 \in K \setminus \{0\}$, where K is the partial-ordering cone of Y , and discussing the EVP in this direction. Very recently, Bednarczuk and Zagrodny (2009) proved an EVP with a closed bounded convex subset D of Y instead of direction k_0. In Chapter 4 we develop this result, considering both Pareto and Kuroiwa's minima of a set-valued map. A corresponding minimal element theorem for a product order is also proved as a underlying fact for the EVP. In Chapter 5 we introduce several new kinds of inferior and superior limits and corresponding kinds of semicontinuity of a set-valued map. Together with the known concepts of semicontinuity, they can be used to have a clearer insight of local behaviors of maps. Then we investigate all major semicontinuity properties of solution maps to a general quasivariational inclusion. Consequences are derived for several particular problems, including some connections to Ekeland's variational principle.

In each chapter comparisons between our results and recent known ones, including even comparisons when applied to particular cases, are provided. Numerous corollaries and examples are also given to illustrate the main results.

I am indebted to many kind people who have significantly contributed to the thesis. First of all, I am deeply grateful to Professor Phan Quoc Khanh, my supervisor, for his kind, continuous guidance and encouragement. My warmest thanks are addressed to Professor Samir Adly, my French cotutelle supervisor, for his fruitful suggestions and helps from a remote distance. I would like to thank very much the University of Science of Hochiminh City and the University of Limoges for accepting me as a cotutelle PhD. student and providing favorable conditions and facilities for my study, and the professors and lecturers who taught me. The last but not least thanks are devoted to my family, friends and especially my colleagues from the seminar of the Section of Optimization and System Theory, headed by Professor Phan Quoc Khanh, especially Dr. Lam Quoc Anh, who has collaborated with me in working on our joint paper on stability of quasivariational inclusions.

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Chapter 5

More kinds of semicontinuity of set-valued maps and stability of quasivariational inclusions

More kinds of semicontinuity of set-valued maps and stability of quasivariational inclusions*

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Abstract. We introduce several new kinds of inferior and superior limits and corresponding kinds of semicontinuity of a set-valued map. Together with the known concepts of semicontinuity, they can be used to have a clearer insight of local behaviors of maps. Then we investigate all major semicontinuity properties of solution maps to a general quasivariational inclusion. Consequences are derived for several particular problems, including some connections to Ekeland's variational principle.

Keywords. Inferior and superior limits, semicontinuity, solution maps, quasivariational inclusions, quasivariational relation problems.

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LIST OF THE PAPERS RELATED TO THE THESIS

1. Khanh P.Q., Quy D.N. (2010), A generalized distance and enhanced Ekeland's variational principle for vector functions, *Nonlinear Analysis* 73, pp. 2245-2259.

2. Khanh P.Q., Quy D.N. (2011), On generalized Ekeland's variational principle and equivalent formulations for set-valued mappings, *Journal of Global Optimization* 49, pp. 381-396.

3. Khanh P.Q., Quy D.N. (2011), On generalized Ekeland's variational principle for approximate Pareto minima of set-valued mappings, *Journal of Optimization Theory and Applications*, accepted.

4. Khanh P.Q., Quy D.N., Some versions of Ekeland's variational principle involving sets of perturbations, submitted for publication.

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